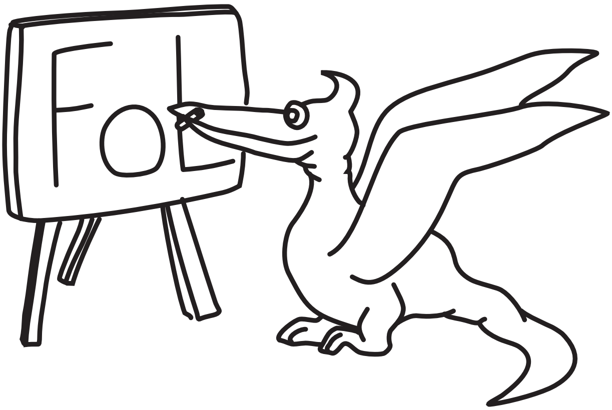
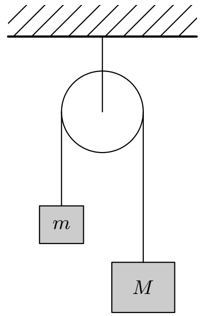
Solutions of 10th Online Physics Brawl



Problem FoL.1 ... (hopefully just) easy pulleys

3 points



Suppose that we have a pulley and some weights hanging from a cord as shown in the figure. Assume that both the cord and the pulley are ideal and massless, $M = 2.0\,\mathrm{kg}$ and $m = 1.0\,\mathrm{kg}$. Find the acceleration of the weight M (in the downwards direction). *Lego wants to discover the simplest pulley problem that half of all teams still fail.*

The cord as well as the pulley are massless (assumption from the problem statement), therefore the forces that the cord exerts on both weights are equal. Let us denote each by T. To describe this, we can write a system of two equations

$$a_1 M = Mg - T\,,$$
$$a_2 m = mg - T\,,$$

where both accelerations are oriented downwards.

We subtract the equations and set $a_1 = -a_2$, because the cord is ideal, which means that it does not change its length. We get

$$a_1 = \frac{M-m}{M+m} g \doteq 3.3\,\mathrm{m\cdot s^{-2}}\,.$$

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Problem FoL.2 ... two circuits

3 points

Legolas found several old identical resistors and connected them in series. Such a circuit had a total resistance $R_\mathrm{S} = 10.0\,\mathrm{k\Omega}$. After that, he decided to connect the resistors in parallel and measured a resistance $R_\mathrm{P} = 1.0\,\Omega$. What was the resistance of a single resistor?

Lego made himself a lego out of resistors.

Let R denote the desired resistance of one resistor and n the total number of the resistors. Then we get

$$R_\mathrm{S} = nR\,,$$
$$R_\mathrm{P} = \frac{R}{n}\,.$$

By multiplying these two equations, we get rid of n and the remaining equation is $R^2 = R_\mathrm{S} R_\mathrm{P}$, from which we obtain the result $R = \sqrt{R_\mathrm{S} R_\mathrm{P}} = 100\,\Omega$. We need $n = 100$ such resistors.

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Problem FoL.3 ... felling trees 3 points

A coniferous tree could be (from a mechanical point of view) approximated by a homogeneous right circular cone with height $h = 40\,\mathrm{m}$ and radius at the base $r = 1.0\,\mathrm{m}$. Find the maximal angle by which its axis may be displaced from the vertical axis before it starts to fall due to its weight. *Dodo was procrastinating on Youtube.*

A rigid body begins to fall when it is displaced in such a way that its centre of mass is no longer straight above its base. The centre of mass of a cone is located at height $h/4$ above the base. The maximal angle at which the tree does not fall yet is the same as the angle between the vertical and the line connecting the centre of mass and the edge of the base. Its magnitude is calculated using the formula $\tan\Phi = \frac{r}{h/4}$, therefore $\Phi = \arctan\frac{4r}{h} \doteq 5.7^\circ$. If the tree is tilted more, it will fall.

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Problem FoL.4 ... the nearest asteroid 3 points

On 16th August 2020, an asteroid (later called 2020 QG) has been recorded as the closest asteroid (spotted so far) that flew by the Earth without colliding with it. At the nearest point of approach, it was only $2\,950\,\mathrm{km}$ above Earth's surface and it had a velocity $v = 12.3\,\mathrm{km{\cdot}s^{-1}}$. How much higher was its velocity compared to the escape velocity at that height above Earth's surface? Find the ratio v/v_{esc}. *Karel made a problem out of news from the website astro.cz.*

The escape velocity is given by the condition that total energy of the object in the gravitational field shall equal zero. Therefore

$$\frac{1}{2}mv^2 - G\frac{Mm}{r} = 0\,,$$

where M denotes the mass of the Earth, G is gravitational constant and v and r is the velocity of the object and its distance from the centre of Earth, respectively. By expressing the velocity we get

$$v_{\mathrm{esc}} = \sqrt{\frac{2GM}{r}} = \sqrt{\frac{2GM}{R+h}}\,,$$

where r was substituted with the sum of radius of Earth and height of the flyby (measured from the ground). If we divide these velocities, we get

$$\frac{v}{v_{\mathrm{esc}}} = v\sqrt{\frac{R+h}{2GM}} \doteq 1.33\,.$$

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Problem FoL.5 ... unstable

3 points

Suppose that we have a cuboid with dimensions $a = 20\,\mathrm{cm}$, $b = 30\,\mathrm{cm}$, $c = 50\,\mathrm{cm}$ *and density* $\rho = 620\,\mathrm{kg{\cdot}m^{-3}}$. *One of its faces with dimensions a and c is lying on a horizontal surface in a homogeneous gravitational field. How stable is it with respect to rotation around one of its bottom edges with length c? Find the smallest amount of energy needed to turn it over. Assume* $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. *Danka's stuff was falling down.*

To overturn the cuboid, we only have to get its center of gravity above the c edge and just barely behind it. This change of the cuboid's position requires an increase in the cuboid's potential energy. This can be counted as the difference in the potential energy of the cuboid's center of gravity. Let's place the zero energy level on the horizontal plane. In the beginning the cuboid has a potential energy of

$$E_{\mathrm{p}1} = \rho abcg\frac{b}{2}\,.$$

When its center of gravity is above the c edge, it will be in h above the plane, where

$$h = \frac{1}{2}\sqrt{a^2 + b^2}\,.$$

Then it will have a potential energy of

$$E_{\mathrm{p}2} = \rho abcgh\,.$$

The stability is the difference of these two energies, therefore

$$\Delta E = E_{\mathrm{p}2} - E_{\mathrm{p}1} = \Delta E = \frac{1}{2}\rho abcg\left(\sqrt{a^2 + b^2} - b\right) \doteq 5.5\,\mathrm{J}\,.$$

The stability of the cuboid is therefore 5.5 J.

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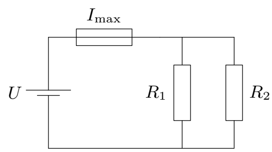
Problem FoL.6 ... safe power

3 points

There are two appliances with resistances $R_1 = 500\,\Omega$ *and* $R_2 = 2\,000\,\Omega$ *respectively, connected in parallel to the terminals of a battery through a fuse with a maximum allowed current* $I_{\max} = 500\,\mathrm{mA}$. *What is the maximum power we could get from the circuit?* *Dodo must pay attention at the dormitory.*



The relation between the voltage and the current is described by the Ohm's law $U = RI$, where R is the total resistance of the connected appliances. The total resistance of the appliances connected in parallel is

$$R = \frac{R_1R_2}{R_1 + R_2} = 400\,\Omega\,.$$

We can calculate the power of an appliance from the voltage on them and the current that flows through them as $P = UI = RI^2$. Maximum power implies maximum current, which is the current that blows the fuse. Plugging in the numerical values, we get $P = 100\,\mathrm{W}$.

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Problem FoL.7 ... the ink-blooded prince

3 points

When Harry Potter did not behave, professor Umbridge would punish him by forcing him to write with his own blood. He needed 1.21 µl of blood on average to write one lowercase letter. A book of school rules has 259 standard pages, each of which contains 1 488 letters on average. However, each thirty-seventh letter is a capital letter, which means it consumes three times more ink than a lowercase letter. Assuming that Harry has 5 l of blood, how many copies of the rules could be made from him? *Jáchym found the whole book series considerably illogical.*

For each 37 letters of the school rules the blood consumption equals writing 39 lower case letters. Let $k = 39/37$ denote that ratio. Let V_a, n_p, n_l and V_H denote the remaining quantities respectively. The volume of the blood required to write on set of school rules is

$$V_\mathrm{b} = n_\mathrm{p} n_\mathrm{l} k V_\mathrm{a} \,.$$

Therefore the number of copies equals

$$n_\mathrm{c} = \frac{V_\mathrm{H}}{n_\mathrm{p} n_\mathrm{l} k V_\mathrm{a}} \doteq 10.2 \,.$$

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Problem FoL.8 ... slowed down train

4 points

While arriving at a station, a train decelerates evenly. Its braking distance is $s = 75\,\mathrm{m}$ and during the penultimate (second to last) second before stopping, it drives a distance $l = 2.25\,\mathrm{m}$. What is its initial velocity v_0 before it begins to brake?
Verča took advantage of a train delay to think of new problems.

The train moves with evenly decelerated motion and during the next to last second it drives the same distance as during the second second of a train accelerating from rest with the same magnitude of acceleration. To make this solution more illustrative, let us mark the beginning and the end of the next to last second as t_1 and t_2 respectively. The distance l could be written as

$$l = \frac{1}{2} a t_2^2 - \frac{1}{2} a t_1^2 = \frac{1}{2} a \left(t_2^2 - t_1^2 \right) \,.$$

We can express acceleration a from this equation as

$$a = \frac{2l}{t_2^2 - t_1^2} \,.$$

We will plug it into the formula for total distance

$$s = \frac{1}{2} a t^2 = \frac{1}{2} \frac{v_0^2}{a} \,,$$

from which we can already express the initial velocity as

$$v_0 = \sqrt{2as} = \sqrt{\frac{4ls}{t_2^2 - t_1^2}} \,.$$

Numerical evaluation for values given in the task results in $v_0 = 15\,\mathrm{m \cdot s^{-1}}$.

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Problem FoL.9 ... we pump oil 4 points

Find the efficiency of a pump, if the electric current flowing through it has an effective value $I = 125\,\mathrm{mA}$ *when attached to a standard power supply of voltage* $U = 230\,\mathrm{V}$*. Our pump pumps oil with density* $\rho = 870\,\mathrm{kg{\cdot}m^{-3}}$ *through pressure difference* $\Delta p = 120\,\mathrm{hPa}$ *with volumetric flow rate* $Q = 0.83\,\mathrm{ml{\cdot}s^{-1}}$. *Dodo wanted to get a rotary vane pump.*

The efficiency η is defined as the ratio between the useful work done (in our case mechanical work against pressure force) and energy supplied (by electric power). The electric network supplies the pump with energy

$$W_1 = UIt\,.$$

in time t. The pump uses it to push the liquid through the given pressure difference, where the useful work is given by the product of the pressure force F_p and the distance s, along which the force is exerted

$$W_2 = F_p s = S\Delta p s = \Delta p V\,,$$

where S is the cross-section of the pump. The distance s is, in fact, the length of the part of the liquid which flowed through the pump in time t, so $V = Ss$ is the volume that flowed through. Plugging it in the equation we obtain

$$\eta = \frac{\Delta p V}{UIt} = \frac{\Delta p Q}{UI} \doteq 0.00035\,,$$

where we used the formula for the flow rate $V = Qt$. The second part of the problem is solvable by dimensional analysis as well.

In this task, we made a numerical error. We decided that the fairest resolution is to not count any points gained or lost because of this problem. We apologize for the troubles.

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Problem FoL.10 ... homogeneous air 4 points

From the point of view of classical optics, every substance is homogeneous. However, today we know that everything consists of particles. Determine the number of particles (molecules) of air that are, under standard conditions, contained in a cube, where the length of one edge of this cube corresponds to the wavelength of the yellow D-line of sodium.

Dodo couldn't sleep again.

The wavelength of the D-line is approximately $\lambda = 590\,\mathrm{nm}$. Let us first determine the mass of the air contained in our cube

$$m = \rho V = \rho \lambda^3\,,$$

where $\rho = 1.29\,\mathrm{kg{\cdot}m^{-3}}$ is the density of the air. We will determine the number of particles using the average molar mass of air $M_\mathrm{m} = 29.0\,\mathrm{g{\cdot}mol^{-1}}$ and the Avogadro constant $N_\mathrm{A} = 6.022\cdot 10^{23}\,\mathrm{mol^{-1}}$ as

$$N = \frac{mN_\mathrm{A}}{M_\mathrm{m}} = \frac{\rho\lambda^3 N_\mathrm{A}}{M_\mathrm{m}} \doteq 5\ 500\ 000\,.$$

Therefore, in the cube where the edge corresponds to the wavelength of the sodium D-line there are millions of molecules.

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Problem FoL.11 ... our old clock

4 points

Our old clock is battery-powered and the capacity of the battery is $E = 1.2\,\mathrm{Wh}$. The hour hand is $r_\mathrm{h} = 5\,\mathrm{cm}$ long and the minute hand is $r_\mathrm{m} = 8\,\mathrm{cm}$ long. The linear density of each hand is $\tau = 10\,\mathrm{g{\cdot}m^{-1}}$. The efficiency of the clockwork is $\eta = 7\,\%$. How long does it take for the clock to stop? The hands rotate continuously and any potential energy released during descending is lost. *Michal was late due to a delayed clock.*

When the watch hands rotate continuously the clockwork does not have to accelerate them. However, it needs to compensate the gravitational force during their movement up. Energy which the clockwork has to spend for one cycle of its hand can be calculated as the difference of the potential energy between the highest and the lowest point. For the minute hand we have

$$\Delta E_\mathrm{m} = mg\Delta h = r_\mathrm{m}\tau g\left(\frac{r_\mathrm{m}}{2} - \frac{-r_\mathrm{m}}{2}\right) = r_\mathrm{m}^2\tau g\,.$$

and for the hour hand

$$\Delta E_\mathrm{h} = r_\mathrm{h}^2\tau g\,.$$

The task does not provide information about the starting position of watch hands so we assume consistent power as follows

$$P = P_\mathrm{m} + P_\mathrm{h} = \frac{\Delta E_\mathrm{m}}{3\,600\,\mathrm{s}} + \frac{\Delta E_\mathrm{h}}{12\cdot 3\,600\,\mathrm{s}} \doteq 1.7\cdot 10^{-7}\mathrm{W} + 5.7\cdot 10^{-9}\mathrm{W} = 1.8\cdot 10^{-7}\mathrm{W}\,.$$

The battery could power the watch for

$$t = \frac{E\eta}{P} \doteq 47\cdot 10^{4}\mathrm{h}\,.$$

Now we can see that it really does not matter that we assumed stable power output of watch, because the period of 12 h is much smaller than the expected battery lifetime. Even with such a low efficiency the watch could work for over 50 years.

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Problem FoL.12 ... cough cough

4 points

A deadly virus with a long incubation period has spread around the world and already infected one ten-thousandth of the total population. Hard-working researchers haven't found any cure yet, but managed to develop a test which can decide whether a given person is infectious or not. It returns the positive result in $99.99\,\%$ of infected cases. However, it returns the false positive result in $0.03\,\%$ healthy cases. It may seem that the test is quite reliable. We have chosen and tested a random person. The test returned a positive value. Estimate the probability that the chosen person is infected. *Matěj felt sick during statistical physics practice.*

Let $p_\mathrm{P} = 0.000\,1$ denote the probability that a random person is infected and $p_\mathrm{N} = 0.999\,9$ the probability that he is healthy. Furthermore let $p_\mathrm{PP} = 0.999\,9$ denote the probability that an infectious person will have positive test and $p_\mathrm{NP} = 0.000\,3$ that a healthy person will have positive test.

The desired probability can be calculated as the ratio between the number of all cases in which is infectious person correctly tested (this equals the product $p_\mathrm{P} p_\mathrm{PP}$)to the number of all people with positive test, which leads to the formula

$$\frac{p_\mathrm{P} p_\mathrm{PP}}{p_\mathrm{P} p_\mathrm{PP} + p_\mathrm{N} p_\mathrm{NP}} = \frac{1}{4} \,.$$

On the other hand, even though the test seemed to be quite reliable, we obtained the result, that a person with a positive test is infectious only with 25 % probability. Do not give up hope…

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Problem FoL.13 … uneven illumination — 4 points

Danka was sitting at a round table with a radius $R = 1.0\,\mathrm{m}$ and noticed that the edge of the table was illuminated much less than the center. What is the difference between illuminance in the center and at the edge of the table if the only light source in the room is a light bulb with luminous intensity $I = 120\,\mathrm{cd}$ hanging $h = 1.5\,\mathrm{m}$ above the center of the table? The light bulb is an isotropic source of light and the ceiling is black. *Danka couldn't see her books well.*

Illuminance E is a photometric quantity defined as the luminous flux incident on a unit area. If we have a point source of light with luminous intensity I at the distance r away from a surface, we can calculate the illuminance as

$$E = \frac{I}{r^2} \cos\alpha \,,$$

where α is the angle between the normal of the surface and the direction of the incident light rays. In the centre of the table, the rays arrive perpendicularly, that is $\cos\alpha = 1$. At the edge of the table however, the rays arrive at an angle α and

$$\cos\alpha = \frac{h}{\sqrt{h^2 + R^2}} \;,.$$

The distance between the edge of the table and the light source is

$$l = \sqrt{R^2 + h^2} \,.$$

The difference of illuminance between the centre and the edge of the table is therefore

$$\delta E = \frac{I}{h^2} - \frac{I}{R^2 + h^2} \frac{h}{\sqrt{h^2 + R^2}} \,,$$

$$\delta E = I \left(\frac{1}{h^2} - \frac{h}{(R^2 + h^2)^{\frac{3}{2}}} \right) \doteq 22.6\,\mathrm{lx} \,.$$

The difference of illuminance between the centre and the edge of the table is 22.6 lx.

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Problem FoL.14 ... autumn in a train 4 points

A train is climbing an icy track uphill. The steepest section of track which the train is still able to climb has a slope angle $\alpha = 1.75^\circ$. When the train gets over the slope, it reaches a station where tracks are horizontal and still icy. What is the shortest distance at which the train can stop from a speed $v = 52\,\mathrm{km{\cdot}h^{-1}}$? *Dodo was waiting for a train.*

During the ascent, it is necessary to have a condition on static friction $F_\mathrm{f} \leq fF_\mathrm{n}$. If only the gravity force F_G affects the train, we use its components in the equations $F_\mathrm{f} = F_\mathrm{G} \sin\alpha$ and $F_\mathrm{n} = F_\mathrm{G}\cos\alpha$. This implies a relation for the slope and the coefficient of friction $\tan\alpha \leq f$.

During the deceleration a, the maximum friction force allowed before the wheels start slipping is given by condition $F_\mathrm{n} = F_\mathrm{G}$ and $F_\mathrm{f} = ma$. Using the condition from above we get $a \leq fg$ where g is gravitational acceleration. The shortest distance at which the train stops is when $a = fg = g\tan\alpha$ and the distance is

$$s = \frac{1}{2}at^2 = \frac{v^2}{2a} = \frac{v^2}{2g\tan\alpha} \doteq 348\,\mathrm{m}\,.$$

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Problem FoL.15 ... the power of a waterfall 4 points

A $h = 30\,\mathrm{m}$ high waterfall has flow rate $Q = 1.2\,\mathrm{m^3{\cdot}s^{-1}}$. Find the total force with which water impacts the ground under the waterfall. Assume that the water quickly flows away from the point of impact and the depth of water under the waterfall is negligible.

Dodo is reminiscing about National park Plitvička jezera.

To solve this problem, we will use Newton's second law in the formulation with momentum

$$F = \frac{\mathrm{d}p}{\mathrm{d}t}\,,$$

which states, that the force is defined as as the time derivative of momentum. In time $\mathrm{d}t$, the river bed decelerates falling water with mass $\mathrm{d}m = \rho\,\mathrm{d}V = \rho Q\,\mathrm{d}t$ from the impact velocity v to zero. We will denote the impact velocity by comparing kinetic and potential energy in a homogeneous gravitational field

$$mgh = \frac{1}{2}mv^2\,, \quad v = \sqrt{2gh}\,,$$

where g is gravitational acceleration. Plugging in, we obtain the force

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{v\,\mathrm{d}m}{\mathrm{d}t} = \frac{v\rho Q\,\mathrm{d}t}{\mathrm{d}t} = \rho Q\sqrt{2hg} \doteq 29\,\mathrm{kN}\,.$$

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Problem FoL.16 ... burning coal 4 points

We burn $m = 213\,\mathrm{mg}$ of pure carbon in a closed vessel of volume $V = 20\,\mathrm{l}$ filled with air. Once the temperature equilibrium between the vessel and its surroundings is restored, we measure the pressure in the vessel. Find the ratio of the pressure after burning to the pressure before burning. *Dodo wanted to be malicious.*

A chemical reaction occurring while burning carbon in an environment with enough oxygen is described by equation

$$\mathrm{C} + \mathrm{O}_2 \longrightarrow \mathrm{CO}_2\,.$$

Since carbon is solid, one mole of gas transforms into one mole of other gas. If we assume that the gas is ideal, the ideal gas law

$$pV = nRT\,,$$

must hold before the reaction as well as after it once the equilibrium is reached again. Consider that the volume V has not changed, nor temperature T (in balance with the surrounding) nor the amount of substance n. Therefore the pressure hasn't changed either. The solution to the task above is $p/p_0 = 1$. We should make sure there is enough oxygen in the vessel. In the case of room conditions, a mole of gas has a volume of approximate $24\,\mathrm{l}$. Therefore there are approximately $0.2\,\mathrm{mol}$ of oxygen molecules. The amount of our carbon is $n = m/M_\mathrm{m} \approx 0.02\,\mathrm{mol}$, which means there is enough oxygen.

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Problem FoL.17 ... heat and phases 4 points

How much higher is the amount of heat required to let ice boil away (Q_b) compared to the amount of heat necessary to only melt it (Q_m)? Find $k = Q_\mathrm{b}/Q_\mathrm{m}$. Our ice is taken from a fridge with inner temperature $t = -18\,{}^\circ\mathrm{C}$. The specific heat capacity of liquid water is $c = 4\,180\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$ and that of ice is $c_0 = 2\,090\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$. The enthalpy of fusion of ice is $l_1 = 334\,\mathrm{kJ\cdot kg^{-1}}$ and the enthalpy of vaporization of water is $l_2 = 2.26\,\mathrm{MJ\cdot kg^{-1}}$. We are interested in the points when the ice fully melts to water (at $0\,{}^\circ\mathrm{C}$) and when the water fully vaporizes (at $100\,{}^\circ\mathrm{C}$). *Karel was wondering about the possibility of scalding above a kettle.*

Let us divide the process into four parts. Q_1 is the heat necessary to warm the ice up to $0\,{}^\circ\mathrm{C}$ (the temperature difference is $\Delta t_1 = 18\,{}^\circ\mathrm{C}$). To calculate it we will use the equation $Q_1 = mc_0\Delta t_1$, where m is the mass of the ice and c_0 is the specific heat capacity of the ice.

Q_2 is the heat necessary to change ice to water at $0\,{}^\circ\mathrm{C}$. We will calculate it using the equation $Q_2 = ml_1$, where we use the enthalpy of fusion (after all, it's about melting).

Q_3 is the heat necessary to warm the water from $0\,{}^\circ\mathrm{C}$ up to $100\,{}^\circ\mathrm{C}$ (the temperature difference is $\Delta t_2 = 100\,{}^\circ\mathrm{C}$). To calculate it, we will use analogous equation as for Q_1, i.e. $Q_3 = mc\Delta t_2$. However, now we used the specific heat capacity of water and a different temperature difference.

Q_4 is the heat necessary to change water to steam at $100\,{}^\circ\mathrm{C}$. The calculation is again analogous to the calculation of Q_2, but now we use the enthalpy of vaporization (after all, this is about vaporization). The equation is therefore $Q_4 = ml_2$.

Now, if we look closely at the situation, we will find that $Q_\mathrm{b} = Q_1 + Q_2 + Q_3 + Q_4$ and that $Q_\mathrm{m} = Q_1 + Q_2$. In total, after canceling out m we have

$$\frac{Q_\mathrm{b}}{Q_\mathrm{m}} = \frac{c_0\Delta t_1 + l_1 + c\Delta t_2 + l_2}{c_0\Delta t_1 + l_1}\,.$$

After numerical substitution we will find out that we have to supply 8.21 times more heat to the ice to boil it away than to melt it.

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Problem FoL.18 ... boiling 4 points

Danka put $V = 1.5\,\mathrm{l}$ of water with a temperature $T_0 = 25\,\mathrm{^\circ C}$ in a pot and set it to boil on a stove with wattage (electric power consumption) $P = 1\,200\,\mathrm{W}$. The efficiency of heating the pot with water (i.e. how much heat is transferred to the pot with water) is $\eta = 0.69$ and the heat capacity of the pot is $C = 500\,\mathrm{J\cdot K^{-1}}$. How long does Danka need to wait till the water starts to boil? Find the necessary constants in tables. *Danka had to wait a long time while cooking.*

We assume the water to have the following properties: density $\rho = 1\,000\,\mathrm{kg\cdot m^{-3}}$, specific heat capacity $c_\mathrm{w} = 4\,180\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$ and boiling point of $T_\mathrm{w} = 100\,\mathrm{^\circ C}$.The heat needed to be absorbed by the pot and the water is

$$Q = V\rho c_\mathrm{w}\left(T_\mathrm{w} - T_0\right) + C\left(T_\mathrm{w} - T_0\right) .$$

The stove transmits this heat in time t and therefore

$$Q = \eta P t \,.$$

Thus

$$\eta P t = \left(T_\mathrm{w} - T_0\right)\left(V\rho c_\mathrm{w} + C\right) \quad .$$

Final formula for the time is

$$t = \frac{\left(T_\mathrm{w} - T_0\right)\left(V\rho c_\mathrm{w} + C\right)}{\eta P} \doteq 613\,\mathrm{s} \approx 10\,\mathrm{min}\,.$$

Danka will wait approximately 10 minutes for the water to start boiling.

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Problem FoL.19 ... internal resistance 4 points

Consider two identical resistors connected in series to a non-ideal voltage source. In this case, the efficiency of the voltage source is 0.87. Find the efficiency of the voltage source, i.e. the ratio between the power consumed by our resistors and the total power supplied by the source, when the two resistors are connected in parallel instead. We assume model the non-ideal voltage source like an ideal voltage source (not a current source) with an internal resistor connected in series. *Matěj wanted to save some money – so he tried saving electricity.*

The efficiency of a power supply source can be calculated as the ratio of the total power to the power on attached resistors. To calculate the power on a resistor, we will use the formula $P = RI^2$, where I is the current flowing through a resistor with resistance R. Let R_i denote the internal resistance of the power supply source and R the total resistance of two resistors in series. Then, for the efficiency of the connection in series, the equation

$$\eta_1 = \frac{RI^2}{(R + R_\mathrm{i})I^2} = \frac{R}{R + R_\mathrm{i}}\,,$$

holds. We used the fact that the current is the same on all devices connected in series. We denote R_i from the equation as

$$R_\mathrm{i} = R\frac{1-\eta_1}{\eta_1}\,.$$

Parallel connection of two resistors can be substitued by one resistor with half resistance. Therefore, two identical resistors connected in parallel have four times smaller resistance compared to serial connection. For the efficiency of the parallel connection we obtain

$$\eta_2 = \frac{\frac{R}{4}I^2}{\left(\frac{R}{4}+R_\mathrm{i}\right)I^2} = \frac{\frac{R}{4}}{\frac{R}{4}+R_\mathrm{i}} = \frac{R}{R+4R_\mathrm{i}} = \frac{R}{R+4R\frac{1-\eta_1}{\eta_1}} = \frac{\eta_1}{4-3\eta_1} \doteq 0.626\,,$$

where we used the formula for R_i.

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Problem FoL.20 ... raft voyage — 4 points

A raft consists of twelve cylinder-shaped wooden logs. Each log is $l = 8.00\,\mathrm{m}$ *long and has a radius* $r = 12.0\,\mathrm{cm}$*. When loaded with* $m = 70\,\mathrm{kg}$ *freight, it floats in such a way that the logs stick* $s = 3.0\,\mathrm{cm}$ *above the water level. What is the density of the wood?*

Jarda was wondering about possible improvements for a trip on water.

The force of gravity, caused by the raft and its freight, must be balanced by the force of buoyancy, which, according to Archimedes' principle, equals the force of gravity of the water displaced by the immersed part of the body

$$F_\mathrm{bu} = V_\mathrm{i}\rho_0 g\,,$$

where ρ_0 is the density of water, g is the gravitational acceleration and V_i is the volume of the immersed part of the raft. For the force of gravity, using the formula for the volume of a cylinder, we have

$$F_G = mg + V\rho g = mg + 12\pi r^2 l\rho g\,,$$

where ρ is the density of the wood. The remaining task is to estimate the immersed volume of the raft. From the geometrical point of view, these are twelve bodies, each of which was made by cutting off a cylinder perpendicularly to its base. The base of each body is a circular segment with the surface S. The formula for the surface can be found in the literature as

$$S = r^2 \arccos\frac{r-h}{r} - (r-h)\sqrt{2hr-h^2}\,,$$

where $h = 2r - s$. After plugging it into the formula for the volume and enumeration we obtain the volume of the immersed part $V_\mathrm{i} = 12Sl \doteq 4.03\,\mathrm{m}^3$.

If we compare the forces mentioned above we can express the density of the wood as

$$\rho = \frac{\rho_0 V_\mathrm{i} - m}{12\pi r^2 l} \doteq 910\,\mathrm{kg{\cdot}m^{-3}}\,.$$

Therefore it is wood with quite high density, but still, we can float on it with a sufficient margin. For the conditions given, a single log with density $\rho \doteq 730\,\mathrm{kg{\cdot}m^{-3}}$ would be enough to carry a person.

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Problem FoL.21 ... mysterious ball

4 points

We have a ball with radius $r = 2.0\,\mathrm{cm}$, which has an unknown mass and is made from an unknown material, hanging from a massless spring. When submerged in water with density $\rho = 1\,000\,\mathrm{kg\cdot m^{-3}}$, the elongation of the spring drops to 80% of its value before the ball is submerged. What is the mass of the ball?

Verča was reminiscing about experiments in high school.

The difference in elongation of the string is caused by the force of buoyancy acting on the submerged ball. The relevant forces can be expressed as

$$0.8F_G = F_G - F_\mathrm{b}\,,$$

where F_G is the force of gravity and F_b the force of buoyancy. Now we need to plug the formulas for forces in and express mass m as

$$0.8mg = mg - V\rho g\,,$$
$$m = \frac{20}{3}\pi r^3 \rho\,.$$

Using the numerical values given we obtain the mass of the ball as $m \doteq 168\,\mathrm{g}$.

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Problem FoL.22 ... illuminated

5 points

Danka was sitting in a futuristic train, which was traveling at a constant speed $v = 1.0\cdot10^5\,\mathrm{km\cdot s^{-1}}$. The train track was perpendicular to a long straight road and along this road, lamps were placed with constant spacing $l = 5.0\,\mathrm{m}$. Danka crossed the intersection of the track and the road. In the landscape's reference frame, all the lamps were turned on at once when the train was at a distance $d_0 = 2.0\,\mathrm{km}$ from the intersection. One lamp is at the intersection, let's give it index 0. What time passes in Danka's reference frame between the moments when light from lamp 0 reaches her and when light from a lamp with index 100 reaches her? Use the exact value of speed of light.

Danka was traveling by train.

The light from the lamp with index 0 reaches Danka in time t_0. In that moment she is in the distance $d_0 + vt_0$ from the crossing. For time t_0

$$t_0 = \frac{d_0}{c - v}\,,$$

holds. Let t_{100} denote the time when light from lamp with index 100 reaches Danka and let r be the distance between the lamp and the position of Danka at time t_{100}. It satisfies

$$r = \sqrt{n^2 l^2 + (d_0 + vt_{100})^2}\,,$$

where $n = 100$. Then

$$t_{100} = \frac{r}{c} = \frac{\sqrt{n^2 l^2 + (d_0 + vt_{100})^2}}{c}\,,$$

holds. This equation rewrites as a quadratic equation for t_{100}, with solutions

$$t_{100} = \frac{vd_0 \pm \sqrt{c^2 {d_0}^2 + (c^2 - v^2)\, n^2 l^2}}{c^2 - v^2}\,.$$

The physically correct one is the positive one. Furthermore, let us calculate the difference $\Delta t = t_{100} - t_0$. We obtain $\Delta t = 2.06 \cdot 10^{-7}\,\mathrm{s}$. However, this time difference is in the reference frame connected with the countryside. Since Danka moves in that reference frame, we need to transform it into the reference frame connected with the train (let us denote it with apostrophe), in which is Danka stationary and her proper time therefore equals the coordinate time t'. We will begin with the transformation formula $t' \to t$

$$t = \gamma \left(\frac{v}{c^2} x' + t' \right)\,,$$

where v is the velocity of one reference frame towards the other, $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ and if Danka has constant x' (coordinate in reference frame connected with the train), for the time difference we obtain

$$\Delta t' = \frac{1}{\gamma} \Delta t \doteq 1.94 \cdot 10^{-7}\,\mathrm{s}\,.$$

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Problem FoL.23 ... a ball

5 points

Anička is playing with a ball of yarn with a radius $R = 5\,\mathrm{cm}$, which is formed by a $l = 100\,\mathrm{m}$ long string. She stands on an inclined plane with a slope angle $\alpha = 3^\circ$. She kicks the ball along the slope upwards, but one end of the string stays stuck at the point where she kicked it away, so the string is unraveling as the ball travels up the slope until the whole ball unravels and only the straight string remains on the slope. Find the smallest initial velocity of the ball needed to reach the state described above. *Matěj read a children's book.*

Let's start with the law of conservation of energy. Once the ball is fully unraveled (i.e. there is only a straight string), its centre of mass is exactly in one half of its length, which is at height $\frac{l}{2} \sin\alpha$. In the beginning, the centre of mass of the ball is at height $R\cos\alpha$. Therefore, the difference in potential energy is $\Delta E = Mg\left(\frac{l}{2}\sin\alpha - R\cos\alpha\right)$. Let v denote the velocity of the kick. The initial kinetic energy of translation is $\frac{1}{2}Mv^2$ and the kinetic energy of rotation is $\frac{1}{2}J\frac{v^2}{R^2}$, where $J = \frac{2}{5}MR^2$. Therefore, we obtain

$$\Delta E = \frac{1}{2}Mv^2 + \frac{1}{5}Mv^2 = \frac{7}{10}Mv^2\,,$$

$$Mg\left(\frac{l}{2}\sin\alpha - R\cos\alpha\right) = \frac{7}{10}Mv^2\,,$$

$$v = \sqrt{\frac{5g}{7}\left(l\sin\alpha - 2R\cos\alpha\right)} \doteq 6.00\,\mathrm{m\cdot s^{-1}}\,.$$

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Problem FoL.24 ... a traveler's problem 5 points

A traveler needs to catch his train and does not have much time. The train station is $s = 1\,000\,\mathrm{m}$ far from him. Unfortunately, he finds himself in the fields, where he has to force his way out with velocity $v_\mathrm{f} = 3.0\,\mathrm{km{\cdot}h^{-1}}$. There is a road leading to the station as well, and on this road, he could jog with an average velocity $v_\mathrm{r} = 7.0\,\mathrm{km{\cdot}h^{-1}}$. However, the road is $l = 600\,\mathrm{m}$ far from him. Find the optimal angle α (measured with respect to a perpendicular to the road) such that the traveler reaches the station as soon as possible if he starts walking towards the road at this angle. *Verča went hiking.*

We will begin with calculation of the time of travel depending on the angle α and other parameters as

$$t = \frac{l}{v_\mathrm{f} \cos \alpha} + \frac{\sqrt{s^2 - l^2} - l \tan \alpha}{v_\mathrm{r}} \,.$$

In order to reach minimal time possible we will differentiate the formula with respect to α and put it equal zero

$$\frac{\mathrm{d}t}{\mathrm{d}\alpha} = \frac{l}{v_\mathrm{f}} \cdot \frac{\sin \alpha}{\cos^2 \alpha} - \frac{l}{v_\mathrm{r}} \cdot \frac{1}{\cos^2 \alpha} = 0 \,.$$

The desired angle expresses as $\alpha = \arcsin\left(v_\mathrm{f}/v_\mathrm{r}\right) \doteq 25.4^\circ$.

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Problem FoL.25 ... a different parallel-plate capacitor 5 points

Consider a capacitor consisting of a square conductive plate with surface area $S = 6.0\,\mathrm{cm^2}$ and a parallel, infinite and grounded conductive plane at a distance $d = 1.1\,\mathrm{mm}$. Find the capacitance of such a capacitor, defined as the ratio of charge on the plate to potential on the plate. *Originally, Vašek prepared a problem with a capacitor with infinite capacitance.*

Both conductive planes create equipotential surfaces. As usual, we choose electrostatic potential φ such that equals zero on the grounded conductor. Let Q denote the electric charge of the square plate and φ_r the electrostatic potential on it. When looking for electrostatic potential in half-space containing the square plate bounded by the infinite plane (let us denote it right half-space), the problem reduces to solving Poisson's equation

$$\Delta\varphi = -\frac{\rho}{\varepsilon_0} \tag{1}$$

in given half-space with boundary condition $\varphi = 0$ on the infinite plane, where ρ is the charge density and ε_0 is vacuum permittivity. This kind of problems is often solved using the *method of images*. We will consider the infinite plane to be a plane of symmetry. If all charges from the right half-space were mirrored to the left half-space with the opposite charge, we would get a new electrostatic problem which solution equals the solution of the original problem in the right half-space. It should be noted that mirroring and change of signs cause the charge on the plane to be zero. Overall it causes the charge density ρ to be antisymmetric when mirrored through the plane of symmetry. Consider that the Poisson equation (1) is linear – therefore potential φ exists and it is antisymmetric when mirrored and just like the charge density ρ is zero on the plane of symmetry. These observations form the basis of the *method of images*.

In our particular problem, the method of images gives electrostatic problem with two identical square plates in the distance $2d$, from which one has potential φ_r and charge Q and the second one has potential $-\varphi_r$ and charge $-Q$. These plates together form a parallel-plate capacitor. Considering that their distance $2d$ is significantly less than their characteristic size $\sqrt{S}$, we may approximate their capacitance as

$$C' = \varepsilon_0 S/2d\,.$$

Now we need to realise, that such capacitor has potential difference twice the difference between plate and the infinite plane and so capacitance, as defined in task, is $C = 2C' = 4.8\,\mathrm{pF}$.

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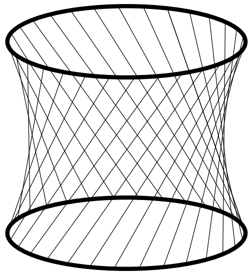
Problem FoL.26 ... oscillating hoop

5 points



Suppose that we have two identical hoops, each with a radius $R = 1.0\,\mathrm{m}$. The upper hoop is in a fixed position, while the lower one is attached to the upper one by several massless cords; each of these cords has the same length $l = 2.0\,\mathrm{m}$ and hangs vertically. Let the mass of the lower hoop be $m = 1.0\,\mathrm{kg}$. If we rotate it a bit around the vertical axis and release, what is its period of small oscillations?

Lego modified a mathematical problem into a physics problem.

As usual, there are plenty of ways how to calculate the period of small oscillations. Here we will use a less well known but a very effective method.

The main idea is to express the energy of our system as a sum of kinetic and potential energy. We will use the angle of displacement from the equilibrium φ as a variable to the relations.

The kinetic energy is simply

$$E_\mathrm{k} = \frac{1}{2} I\omega^2 = \frac{1}{2} mR^2 \dot{\varphi}^2\,.$$

The potential energy is slightly more difficult. For sufficiently small φ we can assume that the cord attachment points have moved from their initial positions by $R\varphi$. Applying the Pythagorean theorem we get the new distance between the hoops $\sqrt{l^2 - R^2\varphi^2}$. If we assume $R\varphi \ll l$, we can approximate the relation as

$$l\sqrt{1 - \left(\frac{R\varphi}{l}\right)^2} \approx l\left(1 - \frac{1}{2}\left(\frac{R\varphi}{l}\right)^2\right) = l - \frac{R^2\varphi^2}{2l}\,.$$

Therefore, compared to the initial position (zero displacement was at the distance l below the upper hoop), the lower hoop is lifted by

$$\Delta h = \frac{R^2\varphi^2}{2l}\,.$$

The difference in potential energy is

$$E_\mathrm{p} = mg\Delta h = \frac{1}{2}\frac{mgR^2}{l}\varphi^2\,.$$

Consider that kinetic and potential energy of simple harmonic oscillator is described by equations

$$E_\mathrm{k} = \frac{1}{2} m \dot{x}^2\,,$$
$$E_\mathrm{p} = \frac{1}{2} k x^2\,.$$

The period is $T = 2\pi\sqrt{m/k}$. Now we just need to notice that the formulas, which we have derived for our system, have the same "pattern" as these for SHO (the only difference is that our variable is angle, not position, but that is not an issue). Let us denote the expressions equivalent to m and k as m_ef and k_ef, respectively, and plug them into the equation for the period of small oscillations. We obtain

$$T = 2\pi\sqrt{\frac{m_\mathrm{ef}}{k_\mathrm{ef}}} = 2\pi\sqrt{\frac{mR^2}{\frac{mgR^2}{l}}} = 2\pi\sqrt{\frac{l}{g}} \doteq 2.84\,\mathrm{s}\,.$$

We probably could have guessed the result straight away...

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Problem FoL.27 ... oscillating 5 points

Suppose that we have a homogeneous square plate with edge length 1 m. *We drill a tiny hole in it, hammer a nail into a wall, hang the plate from the nail through this hole and let the plate oscillate in the vertical plane in which it lies. How far from the centre of the square do we have to drill the hole to maximise the frequency of small oscillations?*

Matěj had some spare metal sheet and didn't know what to do with it.

Let us use the equation for physical (or *compound*) pendulum

$$\omega = \sqrt{\frac{mgl}{J}}\,,$$

where m is the mass of the plate, l is the distance from the axis to the centre of gravity of the plate and J is the moment of inertia of a square with respect to the rotation axis. The moment of inertia can be calculated using the parallel axis theorem, since we know the moment of inertia of a square with respect to the centre of mass $J_0 = \frac{1}{6}ma^2$, where a is the length of its side. For the desired moment of inertia we can write

$$J = J_0 + ml^2 = \frac{1}{6}ma^2 + ml^2\,.$$

We obtain

$$\omega^2 = \frac{gl}{\frac{1}{6}a^2 + l^2}$$

and put the derivative with respect to l equal zero giving

$$g\left(\frac{1}{6}a^2 + l^2\right) - 2gl^2 = 0\,,$$

from which we can calculate the required distance

$$l = \frac{1}{\sqrt{6}} a \doteq 0.408\,\mathrm{m}\,.$$

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Problem FoL.28 ... lego and dice 1.0 — 4 points

We have two small cuboids, one lying on the other – the lower cuboid has mass $M = 2.0\,\mathrm{kg}$, the upper one $m = 1.0\,\mathrm{kg}$. The coefficient of friction between the lower cuboid and the surface under it is 0, the coefficient of static friction between the cuboids is $f_\mathrm{s} = 0.50$ and the coefficient of kinetic friction between them is $f_\mathrm{k} = 0.20$.

Find the magnitude of the force we must exert on the lower cuboid that causes it to move with a constant acceleration $a = 10\,\mathrm{m{\cdot}s^{-2}}$.

Lego was helping his friend with physics, so he at least took some ideas for new problems.

The maximum lateral force that the lower cuboid can impart on the upper is $F_\mathrm{max} = mgf_\mathrm{s} \doteq 5\,\mathrm{N}$. In this case, the upper cuboid would accelerate with the acceleration $a_\mathrm{max} = F_\mathrm{max}/m \doteq 5\,\mathrm{m{\cdot}s^{-2}}$, which is less than the acceleration required by the problem task. Therefore, we are interested in dynamic friction.

The magnitude of the force of friction between those two cuboids is simply $F_\mathrm{k} = mgf_\mathrm{k}$. Therefore, if we denote F to be the force that we impart on the lower cuboid, the resulting acceleration will be equal

$$a = \frac{F - mgf_\mathrm{k}}{M}\,.$$

Now we only need to express $F = aM + mgf_\mathrm{k} \doteq 22\,\mathrm{N}$.

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Problem FoL.29 ... lego and dice 2.0 — 5 points

We have two small cuboids, one lying on the other – the lower cuboid has mass $M = 2.0\,\mathrm{kg}$, the upper one $m = 1.0\,\mathrm{kg}$. The coefficient of friction between the lower cuboid and the surface under it is 0, the coefficient of static friction between the cuboids is $f_\mathrm{s} = 0.50$ and the coefficient of kinetic friction between them is $f_\mathrm{k} = 0.20$.

Find the magnitude of the force we must exert on the lower cuboid that causes it to move with a constant acceleration $a = 1.0\,\mathrm{m{\cdot}s^{-2}}$.

Lego was helping his friend with physics, so he at least took some ideas for new problems.

Consider that the upper cuboid will not accelerate faster than the lower, which implies that the maximal acceleration of the upper cuboid is a. To achieve that, force $am = 1\,\mathrm{N}$ is necessary.

The maximal force that the lower cuboid can impact the upper with is $F_\mathrm{max} = mgf_\mathrm{s} \doteq 5\,\mathrm{N}$, which is more than is necessary, therefore the upper cuboid will accelerate with the acceleration a as well.

Now we need to subtract the force that the lower cuboid imparts on the upper from the force we impart on it. Alternatively, it is enough to realise that these cuboids behave as a single object, which means

$$a = \frac{F}{M+m}\,.$$

Now we will only need to express the force as $F = (M+m)\,a = 3.0\,\mathrm{N}$.

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Problem FoL.30 ... impact accelerator 5 points

We have a large number of point masses in a series in one line. The first point mass has a mass $M_0 = 1\,\mathrm{kg}$ and every subsequent point mass has a mass equal to $70\,\%$ of the previous one. The first point mass starts to move towards the second one with a kinetic energy $E_0 = 50\,\mathrm{J}$. All collisions are perfectly elastic. Which point mass will be the first to have a speed larger than one percent of the speed of light? Neglect any relativistic effects.

Jarda wanted to improve upon CERN technology.

During perfectly elastic collisions, we can use the law of conservation of momentum

$$Mu = Mv_M + mv_m$$

and the law of conservation of energy

$$\frac{1}{2}Mu^2 = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2\,,$$

where M is the mass of the more massive of two points, $m = 0.7M$ is the mass of the less massive point, u is the speed of the more massive point before the collision, v_M is it's speed after the collision, and v_m is the speed of the lighter point after the collision. From the equations we get

$$v_m = \frac{2M}{M+m}u = \frac{2}{1.7}u\,.$$

Apparently, the speed of a point after a collision is simply a multiple of the speed of the previous point. After n collisions, the speed of the last (fastest) point is

$$v_{\max} = \left(\frac{2}{1.7}\right)^n u_0\,,$$

where

$$u_0 = \sqrt{\frac{2E_0}{M_0}} = 10\,\mathrm{m{\cdot}s^{-1}}$$

is the speed of the first mass point before the first collision. Now we plug in $v_{\max}$ and get n as

$$n = \frac{\ln\left(\frac{0.01c}{u_0}\right)}{\ln\left(\frac{2}{1.7}\right)} \doteq 77.6\,.$$

For the index of the first point that surpasses one percent of lightspeed, we need to round up and add one (we get to the nth point after $(n-1)$ collisions), that is 79.

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Problem FoL.31 ... jump on! 5 points

Imagine a sufficiently large cuboid with mass $M = 32.5\,\mathrm{kg}$ *lying on the ground. We throw a smaller cuboid with mass* $m = 11.7\,\mathrm{kg}$ *on it in such a way that right before the impact, the vertical component of its velocity is almost zero, while the horizontal component is* $v = 19.2\,\mathrm{m{\cdot}s^{-1}}$. *The coefficient of friction between the larger cuboid and the ground is* $f_1 = 0.13$, *the coefficient of friction between the two cuboids is* $f_2 = 0.69$. *What is the total distance covered by the larger cuboid?* *Jachym wondered wherever is possible to land with a plane.*

The friction force f_2mg will affect the smaller cuboid until its velocity reaches the velocity of the lower cuboid (let's denote it v_1 and the time when this happens t_1). From the magnitude of this force it follows that the acceleration is $a_2 = -f_2g$.

The smaller cuboid affects the bigger one with the same friction force. The bigger cuboid is also affected by the friction force between it and the ground, which is $f_1(m+M)g$. This force decelerates it, while the friction force between it and the smaller cuboid accelerates it. In total, it's acceleration is

$$a_1 = \frac{f_2mg - f_1(m+M)g}{M}\,,$$

therefore its velocity at t_1 will be $v_1 = a_1t_1$. For the small block, on the other hand, we have $v_1 = v + a_2t_1$; from this equation we can express time as

$$t_1 = \frac{v}{a_1 - a_2} = \frac{Mv}{(f_2 - f_1)(m+M)g}\,.$$

During this time, the bigger cuboid will travel the distance of

$$x_1 = \frac{1}{2}a_1t_1^2\,.$$

During the rest of the trajectory, the cuboids move as one body, so their acceleration is $a_3 = -f_1g$. Deceleration from v_1 to zero will take them

$$t_3 = -\frac{v_1}{a_3} = -\frac{a_1t_1}{a_3}\,.$$

During this time they travel the distance of

$$x_3 = v_1t_3 + \frac{1}{2}a_3t_3^2 = -\frac{1}{2}a_3t_3^2 = -\frac{a_1^2t_1^2}{2a_3}\,.$$

Therefore, overall covered distance will be

$$x = x_1 + x_3 = \frac{1}{2}a_1t_1^2 - \frac{a_1^2t_1^2}{2a_3} = \frac{1}{2}a_1t_1^2\left(1 - \frac{a_1}{a_3}\right) = \frac{mv^2}{2g(m+M)}\frac{f_2m - f_1(m+M)}{f_1(f_2-f_1)(m+M)} \doteq 3.60\,\mathrm{m}\,.$$

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Problem FoL.32 ... disgusting piston 5 points

In a thermally insulated cylinder with an internal cross section $S = 500\,\mathrm{cm}^2$ and a height $l = 50\,\mathrm{cm}$, there is a resistor with resistance $R = 120\,\Omega$. The cylinder is otherwise filled with air at a temperature $T_0 = 20\,{}^\circ\mathrm{C}$ and pressure $p_0 = 101\,\mathrm{kPa}$, and the same kind of air surrounds the cylinder. A current $I = 200\,\mathrm{mA}$ flows through the resistor. A base of the cylinder breaks away when pushed with a force exceeding $F = 500\,\mathrm{N}$. After what time does that happen?

Jarda wanted to break something using air.

For the base to break, the pressure difference must reach

$$\Delta p = \frac{F}{S}\,.$$

At the beginning, both outside and inside the piston, there is the pressure p_0. The pressure inside the cylinder must then rise to $p_0 + \Delta p$, that is $1 + \Delta p/p_0$ times. Since the process will be isochronic, this is also the ratio of the increase in temperature.

The resistor heats the air with the power $P = I^2 R$. All of the energy will be converted to the internal energy of the gas, giving

$$Q = \Delta U\,,$$
$$I^2 R t = m c_V \Delta T\,,$$

where m ist the mass of the gas which we are heating up, c_V is it's specific heat capacity at constant volume which, for air, is approximately $0.72\,\mathrm{kJ{\cdot}kg^{-1}{\cdot}K^{-1}}$, and ΔT is the difference in temperatures at the beginning and the end of the process. We already know that the temperature must increase by the factor $1 + \Delta p/p_0$, that is by $\Delta T = T_0 \Delta p/p_0$.

Now we only need to find the mass of gas in the cylinder, which is simply $m = \rho V = \rho S l$, where $\rho = 1.20\,\mathrm{kg{\cdot}m^{-3}}$ (the density of air in normal conditions). Finally we have

$$t = \frac{\rho S l c_V T_0 F}{S p_0 I^2 R} = \frac{\rho l c_V T_0 F}{p_0 I^2 R} \doteq 130\,\mathrm{s}\,.$$

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Problem FoL.33 ... 4D scouring pad 5 points

Find the resistance between two neighbouring vertices of a four-dimensional cube made of wire. Each edge of the cube has a resistance $R = 1\,000\,\Omega$.

Karel was wondering about multi-dimensional budgeting problems.

A 4D cube can be visualised as two 3D cubes where their relevant vertices are connected. Two neighbouring vertices in the 4D cube are e.g. two vertices neighbouring in the 4D cube but from different 3D cubes. Let us discuss, how could the current flow from one vertex (we will denote it A) into the second one (Z).

The first option, of course, is through the edge connecting them, which has the resistance R. One could see that this trajectory is parallel to the others.

If the current does not flow from A through the edge connecting two 3D cubes, it must flow through the edge of the 3D cube to one vertex of this cube which is neighbour to A. There are three such vertices, and they are interchangeable. Therefore, we will use the often-used trick – imagine, that these vertices are connected perfectly conductively (but still no current will flow through these connections) and let B denote one of them. From A to B there are three resistors with the resistance R in parallel, therefore for the resistance the following will hold (for any n resistors)

$$\frac{1}{R_n} = n \cdot \frac{1}{R} \quad \Rightarrow \quad R_n = \frac{R}{n}\,.$$

For $n = 3$ we obtain that the resistance between A and B equals $R/3$.

In vertex B the current has two possible paths again – through the edge connecting the two 3D cubes or "further" in the original cube. In the first case, it always flows to a vertex neighbouring with Z. We can connect these vertices into one (similarly as we did with B) and denote it Y. Consider that the resistance between Y and Z, as well as the resistance between B and Y, equals $R/3$. In the second case the current has two possible paths (since it can not flow back to A) in each of the three vertices equivalent to B, which means six edges in total. But it will always flow into one of three vertices of the first cube, which is in the distance of two edges from A. As before, we will connect these three vertices into a single one (let us denote it C) and the resistance between B and C is $R/6$.

There are two possibilities in the vertex C. The current may either flow into the second 3D cube – through resistance $R/3$ and end in the vertex X, where the resistance between X and Y equals $R/6$; or it can continue in the first 3D cube – through resistance $R/3$ and it will end in the vertex which is opposite to A – let us denote it D.

From the vertex D the current can flow only into the vertex of the second 3D cube, which is opposite to Z. Let us denote it W. Then the resistance between D and W is R and between W and X again $R/3$. Therefore we can calculate the resistance between C and X as the "direct" (R'_{CX}) path and the path through D and W (R_{CDWX}) in parallel

$$R_{CX} = \frac{R'_{CX} R_{CDWX}}{R'_{CX} + R_{CDWX}} = \frac{\frac{R}{3}\left(\frac{R}{3} + R + \frac{R}{3}\right)}{3\frac{R}{3} + R} = \frac{5}{18}R\,.$$

After that we can similarly calculate the resistance between B and Y as

$$R_{BY} = \frac{R'_{BY} R_{BCXY}}{R'_{BY} + R_{BCXY}} = \frac{\frac{R}{3}\left(\frac{R}{6} + \frac{5}{18}R + \frac{R}{6}\right)}{\frac{R}{3} + \frac{R}{6} + \frac{5}{18}R + \frac{R}{6}} = \frac{11}{51}R\,.$$

Finally we will obtain the resistance between A and Z as

$$R_{AZ} = \frac{R'_{AZ} R_{ABYZ}}{R'_{AZ} + R_{ABYZ}} = \frac{R\left(\frac{R}{3} + \frac{11}{51}R + \frac{R}{3}\right)}{R + \frac{R}{3} + \frac{11}{51}R + \frac{R}{3}} = \frac{15}{32}R \doteq 469\,\Omega\,.$$

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Problem FoL.34 ... levitation by firearm
5 points

Consider a body with mass $M = 12.1\,\mathrm{kg}$, which is kept in the air by sustained fire from AK-47. Calculate the rate of fire necessary for the body to hover and constantly oscillate between two points low above the ground. All the impacts are elastic, the mass of each bullet is $m = 7.93\,\mathrm{g}$ and the bullets are flying straight up with speed $v = 715\,\mathrm{m\cdot s^{-1}}$.

Don't forget to take cover and move only when the enemy is reloading.

Any drop of velocity of the bullets associated with gravity can be neglected due to the low heights involved. We can therefore place the lowest point of the body's oscillatory motion to a height 0. At this height the body is moving downwards with a velocity v_0. In a perfectly elastic collision, both momentum and energy are conserved. If the velocity of the body after a collision is v_0' (this time moving upwards), the laws of conservation can be written as

$$mv - Mv_0 = -mv' + Mv_0' \,,$$
$$\frac{1}{2}mv^2 + \frac{1}{2}Mv_0^2 = \frac{1}{2}mv'^2 + \frac{1}{2}Mv_0'^2 \,,$$

where v' is the downwards velocity of the bullet after the collision. However, we can notice that the mechanical energy is conserved so the velocity v_0 has the same magnitude as v_0'. Substituting $v_0 = v_0'$, the second equation becomes $v^2 = v'^2$. The solution $v = -v'$ does not make any physical sense (it would be as if the collision never happened), so we must have $v = v'$. From the first equation, we obtain

$$v_0 = \frac{m}{M}v \,.$$

Let the fall of the body down from the highest point take a time t, then $v_0 = gt$. The time it takes the body to move up is the same as the time of the fall so we are dealing with a period

$$T = 2t = \frac{2v_0}{g} = \frac{2mv}{Mg} \,.$$

Now we can just calculate the frequency (the rate of fire)

$$f = \frac{1}{T} = \frac{Mg}{2mv} \doteq 10.5\,\mathrm{s^{-1}} \,.$$

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Problem FoL.35 ... burning phosphorus
5 points

Assume that we burn $m = 213\,\mathrm{mg}$ of phosphorus dust in a closed vessel with volume $V = 20\,\mathrm{l}$ filled with air. Once the temperature equilibrium between the vessel and its surroundings is restored, we measure the pressure in the vessel. Find the ratio of the pressure after burning to the pressure before burning the phosphorus. *Dodo wanted to be malicious, v. 2.*

We start from the equation

$$4\mathrm{P} + 5\mathrm{O_2} \longrightarrow 2\mathrm{P_2O_5} \,,$$

i.e. five moles of gas and four moles of phosphorus change into a solid product. The amount of substance of oxygen used in the reaction is therefore

$$n_{\mathrm{O}_2} = \frac{5}{4} n_{\mathrm{P}} = \frac{5m}{4M_{\mathrm{m}}} \doteq 0.008\,59\,\mathrm{mol}\,,$$

where $M_{\mathrm{m}} = 31.0\,\mathrm{g\cdot mol^{-1}}$ is the molar mass of the phosphorus atoms. We will use the ideal gas law in the form of

$$\frac{pV}{Tn} = \mathrm{const}\,,$$

where the amount of substance of the gas and the pressure changes during the examined process. At the beginning there was $n_1 = V/V_{\mathrm{mol}} \doteq 0.892\,9\,\mathrm{mol}$ in the vessel. We obtain the ratio of the pressures as

$$\frac{p_2}{p_1} = \frac{n_2}{n_1} = 1 - \frac{n_{\mathrm{O}_2}}{n_1} = 1 - \frac{5mV_{\mathrm{mol}}}{4M_{\mathrm{m}}V} \doteq 0.990\,4\,.$$

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Problem FoL.36 ... mixed current 4 points

Consider an electronic component which behaves like a resistor with resistance $R = 42\,\Omega$. We let electric current flow through it; this current has an alternating part and a direct part. The alternating part is harmonic with a frequency $f = 50.0\,\mathrm{Hz}$. The minimum value of the current is $12\,\mathrm{mA}$ and the maximum value is $42\,\mathrm{mA}$. The current does not change direction. What is the average power consumed by the component? *Karel wanted to combine.*

Let us begin with the discussion about what is the current like. The problem statement reads, that it has both direct and harmonic components, which can be written as $I = I_{\mathrm{DC}} + I_{\mathrm{AC}} \sin(2\pi f t)$. In general, the formula should contain the phase as well, but since we are interested in the average power, it does not matter at all. Furthermore the problem statement mentiones that the current does not change direction, which means $I_{\mathrm{DC}} > I_{\mathrm{AC}}$. We can write the following system of equations

$$I_{\mathrm{DC}} + I_{\mathrm{AC}} = I_{\max} = 42\,\mathrm{mA}\,,$$
$$I_{\mathrm{DC}} - I_{\mathrm{AC}} = I_{\min} = 12\,\mathrm{mA}\,,$$

from which we can express

$$I_{\mathrm{DC}} = \frac{I_{\max} + I_{\min}}{2} = 27\,\mathrm{mA}\,,$$
$$I_{\mathrm{AC}} = \frac{I_{\max} - I_{\min}}{2} = 15\,\mathrm{mA}\,.$$

We now know what current flows through the appliance. The remaining task is to denote the power being dissipated at time t. Since the power is the product of the current and the voltage, we get

$$P(t) = I(t)U(t) = RI(t)^2 = R\left(I_{\mathrm{DC}}^2 + 2I_{\mathrm{DC}}I_{\mathrm{AC}}\sin(2\pi f t) + I_{\mathrm{AC}}^2 \sin^2(2\pi f t)\right)\,.$$

The power changes periodically with the period $T = 1/f$, therefore if we want to calculate the average power, it is enough to integrate it through one period and divide it by the length of one period

$$\begin{aligned}\overline{P} &= \frac{1}{T}\int_0^T R\left(I_{\mathrm{DC}}^2 + 2I_{\mathrm{DC}}I_{\mathrm{AC}}\sin\left(2\pi ft\right) + I_{AC}^2\sin^2\left(2\pi ft\right)\right)\mathrm{d}t = \\ &= fR\left[tI_{\mathrm{DC}}^2 - 4\pi fI_{\mathrm{DC}}I_{\mathrm{AC}}\cos\left(2\pi ft\right) + I_{\mathrm{AC}}^2\left(\frac{t}{2} - \frac{\sin\left(4\pi ft\right)}{8\pi f}\right)\right]_0^{1/f} = \\ &= R\left(I_{\mathrm{DC}}^2 + \frac{I_{\mathrm{AC}}^2}{2}\right) \doteq 35\,\mathrm{mW}\,.\end{aligned}$$

The evaluation of the integral could be simplified using the fact that the integral of a sine through the whole period is zero and the integral of the square of a sine through an interval equal to a multiple of the half-period is one half of the interval length.

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Problem FoL.37 ... stopping a train

5 points

There is a train with mass $m = 500\,\mathrm{t}$ and speed $v = 100\,\mathrm{km\cdot h^{-1}}$ heading towards a superhero standing on the tracks. The train is at distance $s = 1\,300\,\mathrm{m}$ from the superhero, who wants the train to stop just in front of him. However, the only thing available to him is a laser pointer. What power must his laser have to achieve this feat? The front of the locomotive is a mirror.

Jarda thought up the plot for a new film.

Let's first calculate what the deceleration of the train must be if it stops just shy of the superhero. The velocity of the train is much smaller than the speed of light, so we can consider this to be movement under a constant (negative) acceleration and write

$$s = \frac{1}{2}at^2 = \frac{1}{2}vt\,,$$

i.e. the superhero needs to stop the train in $t = 2s/v$. This means that the train needs to decelerate at a rate of

$$a = \frac{v}{t} = \frac{v^2}{2s}\,.$$

The force that needs to act on the train is

$$F = ma = m\frac{v^2}{2s}\,.$$

Now for the laser. The momentum of a single photon can be expressed as $p = h/\lambda$, where h is Planck's constant and λ is the wavelength of the photon. The energy of the photon is $E = ch/\lambda = cp$, where c is the speed of light (in fact, the momentum is defined as E/c, which is a direct consequence of the formula for relativistic energy and the fact that a photon has no rest mass).

We therefore have a simple relationship between the momentum of photons traveling in the laser and the energy needed for the train to stop. It remains to find the momentum of photons that need to leave the laser per unit time. We can use the well known formula

$$F = \frac{\Delta p}{\Delta t}\,,$$

but here we need to take care. The problem statement says that there is a mirror at the front of the locomotive, i.e. the photons are not absorbed, but rather reflected. The change of momentum the photons undergo upon impact with the train is double the magnitude of their momentum when traveling. Symbolically,

$$m\frac{v^2}{2s} = 2\frac{\Delta p_{\mathrm{laser}}}{\Delta t}\,,$$

where we only substituted for the necessary force.

At the end, we only need to realize that power is energy per unit time

$$P = \frac{\Delta E_{\mathrm{laser}}}{\Delta t} = c\frac{\Delta p_{\mathrm{laser}}}{\Delta t} = cm\frac{v^2}{4s} \doteq 22.2\,\mathrm{TW}$$

We can note that even though lasers in use today can reach powers up to several petawatts, but they are only emitting pulses of light. For continuous lasers, the highest power used tends to be around tens of kilowatts.

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Problem FoL.38 ... solar cube 5 points

Imagine that we have a satellite in the shape of a perfect cube with its surface covered by solar panels. We position it in such a way that it orbits a star with a circular trajectory much larger than the star. The satellite is able to rotate in any way we want. Find the ratio of the maximum to the minimum radiant power absorbed by the satellite. Assume that all incident radiation is absorbed. *Karel saw a solar cube on the Internet.*

At a long distance from the star, light rays are be parallel, i.e. the wavefronts are parallel planes perpendicular to the rays. The power absorbed by the satellite is directly proportional to the cross-sectional area of the satellite (from the point of view of the rays). Therefore, the solution of the problem is the ratio between the areas of maximal and minimal projections of a cube onto a plane.

Let $\mathbf{z}$ be a unit vector in the direction of the radiation. Then the area of the projection satisfies

$$S = \int_{\omega} |\mathbf{z}\cdot \mathrm{d}\mathbf{S}|\,,$$

where ω is the part of the surface of the cube which is exposed to the radiation and $\mathrm{d}\mathbf{S} = \mathbf{n}\,\mathrm{d}S$, where $\mathbf{n}$ denotes the normal vector of an infinitesimal surface $\mathrm{d}S$. We want to calculate that integral for all possible positions of the cube with respect to $\mathbf{z}$, so we can choose the highest and lowest value.

Consider that no more than three faces of the cube can be exposed simultaneously. Also, all exposed faces must have one vertex in common. Let us denote these faces by A, B and C and their normal vectors by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

Furthermore, the dot product of $\mathbf{z}$ and a given normal (mentioned above) is zero for all faces which are not exposed to the radiation. We may claim that there are always exactly three faces exposed, but some of them have normals perpendicular to the rays and therefore do not add anything to the total surface. The set ω from the integral therefore represents the union of faces A, B and C. We obtain

$$S = \int_A |\mathbf{z}\cdot\mathbf{a}|\,\mathrm{d}S + \int_B |\mathbf{z}\cdot\mathbf{b}|\,\mathrm{d}S + \int_C |\mathbf{z}\cdot\mathbf{c}|\,\mathrm{d}S\,.$$

The expression $\mathbf{z}\cdot\mathbf{n}$ is constant on each face. For a cube with unit edges, we get

$$S = |\mathbf{z}\cdot\mathbf{a}| + |\mathbf{z}\cdot\mathbf{b}| + |\mathbf{z}\cdot\mathbf{c}|\ .$$

The vector of the radiation may be divided into three components – each in the direction of one coordinate vector – let us denote them by $\mathbf{z}_a$, $\mathbf{z}_b$ and $\mathbf{z}_c$ respectively. The expression for the surface area simplifies to

$$S = |\mathbf{z}_a|\,|\mathbf{a}| + |\mathbf{z}_b|\,|\mathbf{b}| + |\mathbf{z}_c|\,|\mathbf{c}| = |\mathbf{z}_a| + |\mathbf{z}_b| + |\mathbf{z}_c|\ ,$$

where we used the fact that normal vectors have unit length. The vector of radiation has unit length as well and assuming orthogonality of $\mathbf{z}_a$, $\mathbf{z}_b$ and $\mathbf{z}_c$ (which holds because of orthogonality of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$), we get

$$1 = |\mathbf{z}_a|^2 + |\mathbf{z}_b|^2 + |\mathbf{z}_c|^2\ .$$

Notice that the equation resembles the geometric mean of magnitudes of vectors $\mathbf{z}$. Similarly, the formula for calculation of the surface area can be understood as their arithmetic mean. It is possible to obtain the maximal estimate of the surface area from the inequality between the geometric and arithmetic mean, the minimum can be (with luck) estimated as well.

However, if we do not know the inequality yet, we must continue our work. Let $x = |\mathbf{z}_a|$ and $y = |\mathbf{z}_b|$. Plugging them both into the formula for area, we obtain

$$S = x + y + \sqrt{1 - x^2 - y^2}\,.$$

Let us examine extrema inside intervals of permissible values $x \in \langle 0,\, 1\rangle$, $y \in \left\langle 0,\, \sqrt{1-x^2}\right\rangle$. The partial derivatives of the function $S(x, y)$, which are

$$\frac{\partial S}{\partial x} = 1 - \frac{x}{\sqrt{1-x^2-y^2}}\,,\quad \frac{\partial S}{\partial y} = 1 - \frac{y}{\sqrt{1-x^2-y^2}}\,,$$

must both equal zero. Solving the system of equations results in $x = y$ and then a quadratic equation

$$3x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{3}}$$

giving the only solution. The resulting surface area is $S_1 = \sqrt{3}$.

The second option is to find extrema on the boundaries of the intervals. For $y = 0$ we have

$$S = x + \sqrt{1 - x^2}\,,$$

which is a simple function whose maximum (on the permissible interval for x) is $S_2 = \sqrt{2}$ for $x = \frac{1}{\sqrt{2}}$ and whose minimum is $S_3 = 1$ for $x \in \{0,\, 1\}$.

For the second boundary condition $y = \sqrt{1 - x^2}$, we receive

$$S = x + \sqrt{1 - x^2}\,,$$

which was already examined before.

We have examined all possibilities and obtained all possible candidates for extrema. Since $S_1 > S_2$ we can claim that the maximum of the function $S(x, y)$ on the given interval is $\sqrt{3}$, while the minimum is 1. The desired solution is the ratio of the maximum to the minimum, which equals $\sqrt{3}$.

Finally, it should be pointed out that it is possible to find the solution much easier using intuition – it was enough to imagine a cross-section of the cube when cut by some plane. The minimal value corresponds to the situation when the wavefronts are parallel with one face – the maximal value is reached if the cross-section is a hexagon. However, these are only estimates, not mathematical proofs.

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Problem FoL.39 ... optimal direction 6 points

We have a rigid body with mass $M = 1.0\,\mathrm{kg}$ *lying on a horizontal surface. The coefficient of friction between the body and the surface below it is* $f = 0.4$. *What is the maximum acceleration which the body can reach if we exert a force* $F = 5\,\mathrm{N}$ *on it?* *Lego was moving boxes.*

Assume that the force vector, together with vector in horizontal direction, form the angle φ. Therefore, the resulting net force acting on the body will be horizontal (since the vertical component is compensated by the surface below) and its magnitude will equal

$$F_\mathrm{r} = F\cos\varphi - f\left(Mg - F\sin\varphi\right)\,.$$

Our only freedom is in the choice of angle φ, so we find the derivative of F_v with respect to φ and put the result equal 0.

$$-F\sin\varphi + fF\cos\varphi = 0$$
$$f = \tan\varphi$$

We obtain $\varphi = \arctan f \doteq 0.38$. If we plug the angle into F_r and consider that $a_\mathrm{r} = F_\mathrm{r}/M$, we obtain $a_\mathrm{max} \doteq 1.5\,\mathrm{m{\cdot}s^{-2}}$.

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Problem FoL.40 ... watch out, Danka is throwing 6 points

Danka wants to throw a cricket ball on flat ground. Find the launch angle (measured with respect to the horizontal plane) which she should choose in order to throw the ball as far as possible. Danka is $h = 1.6\,\mathrm{m}$ tall and throws with a velocity $v = 4.5\,\mathrm{m\cdot s^{-1}}$.

Throwing a cricket ball has never been Danka's favourite sport.

Let us choose a coordinate system, where x denotes the horizontal axis and y is the vertical one. Let φ denote the angle under which Danka throws the ball, measured from the horizontal axis. The force, which acts on the ball, acts in $-y$ direction only, therefore the horizontal velocity $v_x = v\cos\varphi$ is constant. The horizontal distance satisfies

$$x = v\cos\varphi\ t\,.$$

The vertical position will change according to the formula for motion with constant acceleration giving

$$y = h + v\sin\varphi\ t - \frac{1}{2}gt^2\,.$$

By elimination of time from these two equations we obtain height y as a function of x

$$y = h + x\tan\varphi - \frac{gx^2}{2v^2\cos^2\varphi}\,.$$

We are interested in the moment when the ball hits the ground, which means $y = 0$. Let d denote the horizontal coordinate of the point of impact. We will use the formula $\cos^{-2}\varphi = \tan^2\varphi + 1$, plug it to the equation and obtain

$$0 = h + d\tan\varphi - \frac{gd^2}{2v^2} - \frac{gd^2\tan^2\varphi}{2v^2}\,.$$

This is quadratic equation not only in d, but also in $\tan\varphi$. Let us write down the variant where $\tan\varphi$ is the desired unknown variable

$$-\tan^2\varphi\frac{gd^2}{2v^2} + d\tan\varphi - \frac{gd^2}{2v^2} + h = 0\,.$$

Now a number of tiny and simple physics ideas. For most of possible values of d we will find two angles that lead to the distance given. If we increase d, the difference between these two angles approaches zero – for $d = d_{\max}$ there is only one angle which leads to such distance. In terms of mathematics, the discriminant equals zero. Let us calculate it from the quadratic equation for $\tan\varphi$ and put it equal 0. This leads to

$$d_{\max}^2 + 4\frac{gd_{\max}^2}{2v^2}\left(h - \frac{gd_{\max}^2}{2v^2}\right) = 0\,,$$

from which we express $d_{\max}$

$$d_{\max} = \frac{v}{g}\sqrt{2gh + v^2}\,.$$

Now we continue solving the quadratic equation for $\tan\varphi$, which has reduced to

$$\tan\varphi = \frac{v^2}{gd}\,.$$

Plugging in $d = d_{\max}$ we obtain

$$\tan\varphi = \frac{v}{\sqrt{2gh + v^2}} = 0.63\,.$$

The desired optimal angle is 32°. Consider that for $h > 0$ the optimal angle is always smaller than 45° and with increasing h it approaches zero.

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Problem FoL.41 ... laser at a dormitory

4 points

Jáchym shines a laser from a dormitory window towards the ground. Matěj measures the radiation of the laser on the ground, $\Delta h = 42.0\,\mathrm{m}$ below Jáchym. The laser shines with a frequency f_0 but Matěj measures a different frequency f. What is the ratio $|f - f_0|\,/f_0$?

Karel repeatedly heard about a black hole being observed.

A photon with frequency f has energy $E = hf$, where h is the Planck's constant. As the photon moves down in the gravitational field, it loses its potential energy and $\Delta E = mg\Delta h$, where m is the "mass" of the photon. While it may seem strange to talk about mass of a photon (which has zero rest-mass), we know from relativity that $E = mc^2$ and therefore

$$m = \frac{hf}{c^2} \approx \frac{hf_0}{c^2}\,.$$

A change in energy therefore leads to a change in frequency. The change of potentional energy is

$$E - E_0 = \Delta E \quad\Rightarrow\quad h\,(f - f_0) = mg\Delta h \approx \frac{hg\Delta h}{c^2} f_0\,.$$

Finally we obtain the required ratio as

$$k = \frac{|f - f_0|}{f_0} \approx \frac{g\Delta h}{c^2} \doteq 4.58\cdot 10^{-15}\,.$$

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Problem FoL.42 ... pyramid

6 points

Imagine a wooden (density $\rho_\mathrm{w} = 600\,\mathrm{kg\cdot m^{-3}}$) symmetrical square-base pyramid floating on the surface of a calm lake. A part of the pyramid with height $h = 6.0\,\mathrm{cm}$ is under the water surface. We impact the pyramid in the vertical direction. Find the frequency of small oscillations of the pyramid. The density of the water is $\rho_\mathrm{l} = 1\,000\,\mathrm{kg\cdot m^{-3}}$. Assume that these oscillations do not affect the height of water level in the lake. The pyramid has an ideal shape, it is homogeneous and pointing downwards (i.e. the apex is lower than the base).

Vítek wondered about global warming.

Firstly we will choose the coordinate system such that the origin is in the apex of the pyramid when the pyramid remains at rest. Next, before we make the impact, the system is in equilibrium, which means that buoyant force equals the weight. If s denotes the surface of a cross-section at the height h and S denotes the surface of the base at the height H (where H is the height of the whole pyramid) and if we consider that the weight of the pyramid equals $M = 1/3\rho_\mathrm{w}SH$, we get

$$sh\rho_\mathrm{l} = SH\rho_\mathrm{w}\,. \tag{2}$$

If we look at a vertical cross-section along the pyramid altitude, looking the angles between the vertical altitude and the horizontal base we get

$$\frac{h}{a} = \frac{H}{A}\,,\,\frac{s}{S} = \frac{h^2}{H^2}\,,$$

where A and a denote the length of the base at given heights. Using this relation and (2) we can find the equation

$$h^3\rho_\mathrm{l} = \rho_\mathrm{w}H^3\,.$$

The motion of the pyramid after that small vertical impact could be described using equation

$$M\ddot{z} = -\rho_\mathrm{l}g\,\mathrm{d}V = -\rho_\mathrm{l}gsz\,,$$

where $\mathrm{d}V \approx sz$ for sufficiently small z is infinitesimal volume difference of the part which is under water in addition (compared to the system in equilibrium). Plugging in for the mass we get

$$\frac{1}{3}\rho_\mathrm{w}SH\ddot{z} = -\rho_\mathrm{l}gsz\,,$$

$$\ddot{z} + \frac{3g\rho_\mathrm{l}h^2}{\rho_\mathrm{w}H^3}z = 0\,.$$

The equation above is the equation of a simple harmonic oscillator, therefore for the frequency of small oscillations, we can write

$$\Omega^2 = \frac{3g\rho_\mathrm{l}h^2}{\rho_\mathrm{w}H^3} = \frac{3g}{h}\,,$$

$$f = \frac{1}{2\pi}\sqrt{\frac{3g}{h}}\,.$$

Using numerical values from the problem statement, we obtain the desired frequency $f \doteq 3.52\,\mathrm{Hz}$.

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Problem FoL.43 ... pole dance 6 points

We have two coaxial cylinders – the smaller one has an outer radius $r = 1.0\,\mathrm{cm}$ *and the bigger one has an inner radius* $R = 10\,\mathrm{cm}$*. The cylinders are connected by a rod positioned in the radial direction. On the rod, next to the surface of the smaller cylinder, there is a small bead. There is one more small ball on the surface of the smaller cylinder. We spin the whole system as a rigid body around the common axis with an angular velocity* $\omega = 2.0\,\mathrm{rad{\cdot}s^{-1}}$ *and then release the bead and the ball simultaneously. The bead can only move on the rod; the ball can move freely. What is the absolute difference between the times when the ball and the bead hit the inner surface of the larger cylinder?* *Jarda got a bit flummoxed from this.*

We will calculate the motion of the ball (the one which is not on the rod) first. If we assume both cylinders rotate with the angular velocity ω, the ball will move with the velocity $r\omega$ in the direction tangential to the smaller cylinder. When we release the ball, it will not change its velocity and move in a straight line until it hits the larger cylinder. The question is how long is the straight line. We obtain its length using the Pythagorean theorem as $s_1 = \sqrt{R^2 - r^2}$. Therefore, the time between the release of the ball and the moment when it hits the larger cylinder is

$$t_1 = \frac{\sqrt{R^2 - r^2}}{r\omega} \doteq 5.0\,\mathrm{s}\,.$$

Let us investigate the bead on the rod now. Since it is on the rod, it moves with the constant angular velocity ω even after its release from the cylinder. We will be using the reference frame connected to the rotating cylinders (and most importantly, with the rod). In this reference frame, the bead will be acted on by the centrifugal force, which is given as $F_{\mathrm{cen}} = m\omega^2 r(t)$, where $r(t)$ denotes the position of the bead on the rod in time t. By dividing the formula by m we get the acceleration of the bead as a function of time t. Consider that the acceleration is the second derivative of $r(t)$, so we get the differential equation

$$\ddot{r}(t) = \omega^2 r(t)\,.$$

Similar equations are most easily solved using the so-called characteristic polynomial. This means that we assume that the solution is exponential function $r(t) = r_0 \mathrm{e}^{\lambda t}$. We plug it into the equation and get

$$\lambda^2 r_0 \mathrm{e}^{\lambda t} = \omega^2 r_0 \mathrm{e}^{\lambda t}\,.$$

First of all, consider that $r_0 = 0$ would satisfy the equation. This makes sense – if the bead's initial position was on the axis, there would not be any force to push it. However, its initial position is on the smaller cylinder and therefore such a solution is not interesting for us. We also know, that an exponential is never zero, therefore we can cancel it out. We cancel out r_0 as well. From the remaining equation, we obtain $\lambda = \pm\omega$. If we solve the differential equation of order n, we get n solutions. The next step is to write the solution as sum of individual solutions. In our case

$$r(t) = r_1 \mathrm{e}^{\omega t} + r_2 \mathrm{e}^{-\omega t}\,.$$

This is general solution of the motion of the bead on the rod rotating with the angular velocity ω. The remaining task is to investigate how will it move in our case. We know that the position at time $t = 0$ is r and the velocity at time $t = 0$ is zero. We get a system of equations

$$\begin{aligned} r &= r_1 + r_2 \\ 0 &= \omega r_1 - \omega r_2\,, \end{aligned}$$

which holds for $r_1 = r_2 = r/2$. Plugging this to the general solution, we get

$$r(t) = \frac{r}{2}\left(\mathrm{e}^{\omega t} + \mathrm{e}^{-\omega t}\right) = r\cosh\omega t\,.$$

If we had not noticed the cosh, it would not have been a mistake, but then we would have to find the resulting time somehow else – using the substitution $x = \mathrm{e}^{\omega t}$ – we would get a quadratic equation or using a numerical calculation.

The remaining task is to find the time when the bead hits the larger cylinder. Mathematically said, for which t_2 the formula $R = r(t_2)$ holds? From the formula above we obtain

$$t_2 = \frac{1}{\omega}\operatorname{argcosh}\frac{R}{r} \doteq 1.5\,\mathrm{s}\,.$$

Subtracting the times we get the desired time difference between the impacts of the ball and the bead $\Delta t = t_1 - t_2 \doteq 3.5\,\mathrm{s}$.

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Problem FoL.44 . . . a fall into the unknown 7 points

We have an infinite grounded conductive plane in free space without any gravitational fields. At a distance $d = 1.00\,\mathrm{m}$ *from the plane, there is a small ball with mass* $m = 2.00\,\mathrm{g}$ *and electric charge with magnitude* $q = 4.00\,\mathrm{\mu C}$*. We release the ball. How long does it take for the charge to drop on the plane?*

Jarda knew that it would fall eventually, but he wanted to know when exactly.

Firstly, we want to find the force that the plane exerts on the charge. A charge with the opposite sign and the same magnitude is electrostatically induced on the plane. In the equilibrium state, the charges do not move inside the plane, therefore no force is exerted on them, the potential on the plane is constant and since we are free to add a constant to potential, we choose the potential to be zero on the plane. The net potential is the sum of the potential φ_p due to charges of the plane and the potential φ_q due to the charge q. Therefore

$$\varphi_q + \varphi_\mathrm{p} = 0\,,$$
$$\varphi_\mathrm{p} = -\varphi_q\,.$$

The potential on the plane is the same as if there was an opposite charge $-q$ on the opposite side of the plane at the same distance d. Now we can substitute all induced charges on the plane by a single charge $-q$. The force that the charge q exerts on $-q$ is

$$F = \frac{1}{4\pi\varepsilon}\frac{q^2}{(d+d)^2} = \frac{1}{16\pi\varepsilon}\frac{q^2}{d^2}\,.$$

Because the charges have opposite signs, the charge q is attracted to the plane. This calculation is based on the method of image charges. We can see that the force is inversely proportional to the square of the distance from the plane to the charge. When we have utilized the plane as a mirror, why not utilise motion of planets too? Gravitational force is, similarly to our force F, proportional to r^{-2}. The motion of the charge, therefore, satisfies Kepler's laws of planetary motion with changed constants. Kepler's third law is

$$\frac{a^3}{T^2} = \frac{MG}{4\pi^2}\,.$$

We need to use our constant instead of a constant derived from Newton's gravitational law, the physical behaviour of the system remains unchanged otherwise. The equation becomes

$$\frac{a^3}{T^2} = \frac{q^2}{64\pi^3\varepsilon m}.$$

What is this equation good for? The charge's trajectory will not be an ellipse, but a straight line perpendicular to the plane. However, if we assume that it is an infinitely thin ellipse, with its foci almost in the original and final point of the trajectory, then such an ellipse approximates the straight line segment on which the charge moves. This ellipse has a semi-major axis a with length $\frac{d}{2}$. The period equals

$$T = \sqrt{\frac{64\pi^3 \varepsilon m \left(\frac{d}{2}\right)^3}{q^2}} = \sqrt{\frac{8\pi^3 \varepsilon m d^3}{q^2}}\,.$$

The move from the initial point to the plane lasts only half a period, therefore the time, which is the final solution to this problem, equals

$$t = \frac{T}{2} = \sqrt{\frac{2\pi^3 \varepsilon m d^3}{q^2}} \doteq 0.262\,\mathrm{s}\,.$$

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Problem FoL.45 ... spring of knowledge 7 points

Jáchym owns a well of knowledge, which has depth $h = 32\,\mathrm{m}$ and a constant circular cross-section with radius $r = 1.5\,\mathrm{m}$. At the bottom of the well, in the middle, there is a point which radiates knowledge uniformly into the whole space around it with power P_0. Knowledge propagates in space similarly to light rays. The reflectance of the sides of the well is $k = 0.42$. The bottom of the well of knowledge does not reflect anything, it absorbs knowledge perfectly. Let P denote the total knowledge power radiated out of the well. Find the ratio P/P_0.

Wells are inexhaustible sources of ideas.

The problem has radial symmetry. Imagine one of the possible two-dimensional cross-sections – the well is projected into a rectangle with dimensions $2r \times h$, while the source of knowledge is in the middle of the bottom side. Let φ denote the angle between a knowledge ray and the vertical. Then all rays with angles from $\varphi_0 = 0$ to

$$\varphi_1 = \arctan \frac{r}{h}$$

exit the well without any reflection and therefore with their original intensity (power). In general, rays with the angle between φ_i and φ_{i+1}, where

$$\varphi_i = \arctan \frac{(2i-1)\,r}{h}\,,$$

reflect i-times in total (let us call them rays in the i-th zone). Their intensity is k^i times the original intensity.

In order to solve the problem, we need to find out what power corresponds to each zone. Imagine a ball with a radius R and with its center in the source of knowledge. The flux of knowledge is uniform on its surface and has the magnitude

$$I_0 = \frac{P_0}{4\pi R^2}\,,$$

which is the total power divided by the surface area. The power in each zone equals $I_0 S_i$, where S_i is the part of the ball's surface coresponding to the zone between angles φ_i and φ_{i+1}. We can calculate it as

$$S_i = \int_{\varphi_i}^{\varphi_{i+1}} 2\pi R \sin\varphi R \,\mathrm{d}\varphi = 2\pi R^2 \left[-\cos\varphi\right]_{\varphi_i}^{\varphi_{i+1}} = 2\pi R^2 \left(\cos\varphi_i - \cos\varphi_{i+1}\right) .$$

The total power leaving the well in the i-th zone is

$$p_i = k^i I_0 S_i = P_0 \frac{k^i}{2} \left(\cos\varphi_i - \cos\varphi_{i+1}\right) ,$$

while the desired ratio of powers is the sum

$$\eta = \frac{P}{P_0} = \sum_{i=0}^{\infty} \frac{p_i}{P_0} = \frac{1}{2} \left(1 - \frac{(1-k)}{k} \sum_{i=1}^{\infty} k^i \cos\varphi_i \right) .$$

Using the trigonometric identity

$$\cos\arctan x = \left(1 + x^2\right)^{-\frac{1}{2}}$$

we can modify the formula to

$$\eta = \frac{1}{2} \left(1 - \frac{(1-k)}{k} \sum_{i=1}^{\infty} k^i \left(1 + \left(\frac{(2i-1)\,r}{h}\right)^2\right)^{-\frac{1}{2}} \right) .$$

The sum must be calculated numerically; it is approximately 0.716. The ratio of power radiated out of the well to total radiated power is $\eta \doteq 5.53 \cdot 10^{-3}$.

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Problem FoL.46 ... an incredible space battle 8 points

Imagine us in the middle of a fight in space between civilization 1 and civilization 2. A battle cruiser of civilization 1 has just launched a rocket on a battle cruiser of civilization 2. The cruisers are $s = 5.00\,\mathrm{km}$ from each other and do not move with respect to each other. The properties of the rocket are: initial mass $m_0 = 5.00\,\mathrm{t}$, engine thrust $T = 1.50 \cdot 10^5\,\mathrm{N}$, specific impulse (exhaust velocity with respect to the rocket) $u = 3.00\,\mathrm{km{\cdot}s^{-1}}$. The engines are set to maximum thrust from launch till the moment of impact. Find the velocity of the rocket right before it crashes into the cruiser of civilization 2. *Jindra was watching Star Wars.*

The velocity of the rocket is described by the Tsiolkovsky equation

$$v(t) - v_0 = u \ln \frac{m_0}{m(t)} . \tag{3}$$

The initial velocity v_0 is zero and the rocket's mass depends on time according to the formula $m(t) = m_0 - Rt$, where R is the fuel mass flow rate

$$R = \frac{T}{u} .$$

Numerically, it is $R = 50.0\,\mathrm{kg{\cdot}s^{-1}}$. Substituting it into the equation (3), we get

$$v(t) = u \ln \frac{m_0}{m_0 - Rt}\,. \tag{4}$$

By integrating the equation (4) we obtain the formula for the distance covered at a given time. The initial condition is $s(t = 0) = 0$, therefore

$$s(t) = ut + ut \ln \frac{m_0}{m_0 - Rt} - \frac{m_0 u}{R} \ln \frac{m_0}{m_0 - Rt} \tag{5}$$

After numerically solving the equation (5), we get the time when the rocket reaches the distance $s = 5.00\,\mathrm{km}$. Let's plug it into the equation (4) and calculate the rocket's velocity at the point of collision. We get $t \doteq 17.7\,\mathrm{s}$ and $v \doteq 584\,\mathrm{m{\cdot}s^{-1}}$. We can tell that the warring civilizations are not very advanced, since their rockets move very slowly compared to the speed of light. Therefore, we could calculate the solution safely without using the special theory of relativity.

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Problem FoL.47 ... rotational pump 8 points

Vítek would like to get some water from his well, but he does not want to keep pulling the bucket up. Therefore, he gradually stirred the water around faster and faster until, at an angular velocity $\omega = 11\,\mathrm{rad{\cdot}s^{-1}}$, the water started flowing out of the well all by itself. Vítek knows the depth of the well (from the top edge to the ground at the bottom) $h = 47\,\mathrm{m}$. The well has a circular cross-section with a radius $r_0 = 1.6\,\mathrm{m}$. What was the height of the water column (from the bottom of the well to the water surface) before Vítek started spinning the water around? Jáchym heard that problems about wells were popular in Fyziklání 2020.

Let's introduce cylindrical coordinates in which we label the horizontal radial distance from the axis of the well as r, an angle of rotation with respect to the axis of the well as φ and the height above the bottom as z.

The forces acting on a small volume of rotating water with mass m and at distance r from the axis are centrifugal force $F_\mathrm{o} = m\omega^2 r$ and force of gravity $F_g = mg$, plus some buoyant hydrostatic forces. The surface is nothing else than a region with a constant potential energy. The potential energy due to the centrifugal force can be calculated as

$$E_\mathrm{c}(r) = \int_0^r -F_\mathrm{c}(x)\,\mathrm{d}x = -\int_0^r m\omega^2 x\,\mathrm{d}x = -\frac{1}{2} m\omega^2 r^2\,.$$

The minus sign comes from the fact that we are integrating against the force which we would need to exert to counteract the centrifugal force. The potential energy due to the force of gravity is

$$E_g(z) = \int_0^z F_g\,\mathrm{d}x = mgz\,.$$

As we said, the water surface is a surface with constant potential. (There are also no buoyant forces on the surface.) For every point on the surface with coordinates r_s, z_s, the following holds

$$E_\mathrm{c}(r_\mathrm{s}) + E_g(z_\mathrm{s}) = \mathrm{const}\,,$$

From this condition, we obtain the height of the surface as a function of radial distance

$$z_{\mathrm{s}}(r) = \frac{\omega^2}{2g} r^2 + z_0 \,,$$

where z_0 is the height of the water surface in the center of the well. Of course, this only holds if the surface is always above the bottom of the well. If there wasn't enough water in the well, the parabola we obtained could intersect the bottom of the well, but for now, let's assume that this is not the case. We can later check whether this assumption holds.

The volume of the water does not change when it is spinning. THe initial volume was $V = \pi r_0^2 z_{\mathrm{v}}$, where z_{v} is the original height of the water column. After spinning, we have

$$V = \int_0^{r_0} \int_0^{2\pi} z_{\mathrm{s}}(r) r \,\mathrm{d}\varphi \,\mathrm{d}r = \int_0^{r_0} 2\pi z_{\mathrm{s}}(r) r \,\mathrm{d}r = 2\pi \int_0^{r_0} \left(\frac{\omega^2}{2g} r^3 + z_0 r \right) \mathrm{d}r = \pi \left(\frac{\omega^2}{4g} r_0^4 + z_0 r_0^2 \right) .$$

From this, we can express the height of water in the center

$$z_0 = z_{\mathrm{w}} - \frac{\omega^2}{4g} r_0^2 \,.$$

We know from the problem statement that at the given angular velocity, the water just started to flow out of the well. We can express that as $z_{\mathrm{s}}(r_0) = h$. From this condition, we get

$$h = \frac{\omega^2}{2g} r_0^2 + z_0 = \frac{\omega^2}{2g} r_0^2 + z_{\mathrm{w}} - \frac{\omega^2}{4g} r_0^2 \,.$$

The original height of the water in the well that we are looking for is then

$$z_{\mathrm{w}} = h - \frac{\omega^2}{4g} r_0^2 \doteq 39\,\mathrm{m} \,.$$

At the end, we just check that in this case $z_0 \doteq 31\,\mathrm{m}$, which means our assumption was correct.

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Problem FoL.48 ... ideal centrifuge 9 points

Imagine a tube filled with ideal gas with molar mass $M_m = 36\,\mathrm{g{\cdot}mol^{-1}}$. The length of the tube is $r_0 = 1.00\,\mathrm{m}$, its thickness is negligible compared to its length. We spin the tube around the axis perpendicular to the tube and passing through one of its bases, with an angular velocity $\omega = 451.00\,\mathrm{s^{-1}}$. The gas inside will settle in equilibrium, at a temperature $T = 300\,\mathrm{K}$ along with the tube. Find the distance between the axis of rotation and the centre of mass of the air in the tube. The tube is rotating in a horizontal plane. *Jáchym wanted to separate air.*

Let us choose a coordinate describing distance from the axis of rotation up to the end of the tube, i.e. from 0 to r_0, denoted by r. In some section $\mathrm{d}r$, we have gas with mass $\mathrm{d}m = \lambda(r)\,\mathrm{d}r$. When the equilibrium is reached, the temperature in the tube must be the same as at the beginning (since the tube is thermally isolated, the process must be adiabatic). From the equation of state, we express the pressure in a given section

$$p = \frac{\mathrm{d}nRT}{\mathrm{d}V} = \frac{\mathrm{d}nRT}{S\,\mathrm{d}r} = \frac{\mathrm{d}mRT}{M_{\mathrm{m}} S\,\mathrm{d}r} = \frac{RT}{M_{\mathrm{m}} S} \lambda \,,$$

where S is the cross-sectional area of the tube. From this formula, we get

$$\mathrm{d}p = \frac{RT}{M_\mathrm{m} S} \lambda' \,\mathrm{d}r$$

There must be a centripetal force

$$\mathrm{d}F_\mathrm{d} = \mathrm{d}m\omega^2 r\,,$$

exerted on a given section of the tube. This force must be caused by difference in pressure. Therefore $S\,\mathrm{d}p = \mathrm{d}F_\mathrm{d}$ holds. Putting it together, we get

$$\frac{RT}{M_\mathrm{m}} \lambda' \,\mathrm{d}r = \mathrm{d}m\omega^2 r\,,$$

which is a differential equation

$$\lambda' = \frac{\omega^2 M_\mathrm{m}}{RT} \lambda r = 2k^2 \lambda r\,,$$

where k is a wisely-defined constant

$$k = \sqrt{\frac{\omega^2 M_\mathrm{m}}{2RT}} \doteq 1.212\,\mathrm{m}^{-1}\,.$$

The solution of the equation is

$$\lambda = A\mathrm{e}^{k^2 r^2}\,,$$

where A is a constant. We calculate the center of mass

$$T_r = \frac{1}{m} \int_0^{r_0} r\,\mathrm{d}m = \frac{\int_0^{r_0} r\lambda\,\mathrm{d}r}{\int_0^{r_0} \lambda\,\mathrm{d}r} = \frac{A\int_0^{r_0} r\mathrm{e}^{k^2r^2}\,\mathrm{d}r}{A\int_0^{r_0} \mathrm{e}^{k^2r^2}\,\mathrm{d}r}$$

and substituting $x = kr$, we get the result

$$T_r = \frac{1}{2k} \frac{\int_0^{kr_0} 2x\mathrm{e}^{x^2}\,\mathrm{d}x}{\int_0^{kr_0} \mathrm{e}^{x^2}\,\mathrm{d}x} = \frac{1}{2k} \frac{\left[\mathrm{e}^{x^2}\right]_0^{kr_0}}{\int_0^{kr_0} \mathrm{e}^{x^2}\,\mathrm{d}x} = \frac{1}{2k} \frac{\mathrm{e}^{k^2r_0^2} - 1}{\int_0^{kr_0} \mathrm{e}^{x^2}\,\mathrm{d}x} = \frac{1}{k} \frac{\mathrm{e}^{k^2r_0^2} - 1}{\sqrt{\pi}\,\mathrm{erfi}(kr_0)} \doteq 0.629\,\mathrm{m}\,,$$

where erfi is the imaginary error function.

Statistical solution

It is possible to solve the problem using statistical physics as well, but the ability to work with statistical ensembles is required.

The one-particle Hamiltonian is

$$H_1(X_1) = \frac{\mathbf{p}^2}{2m} - \boldsymbol{\omega} \cdot \mathbf{L} = \frac{\mathbf{p}^2}{2m} - \omega \mathbf{n}_3 \cdot \mathbf{x} \times \mathbf{p} = \frac{\mathbf{p}^2 - 2m\omega \mathbf{p} \cdot \mathbf{n}_3 \times \mathbf{x}}{2m} =$$
$$= \frac{(\mathbf{p} - m\omega \mathbf{n}_3 \times \mathbf{x})^2}{2m} - \frac{m\omega^2}{2} (\mathbf{n}_3 \times \mathbf{x})^2 = \frac{\mathbf{p}'^2}{2m} - \frac{m\omega^2}{2} r^2\,,$$

where X_1 is microstate of one particle, m is its mass and vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{p} = (p_1, p_2, p_3)$ represent its coordinates and momentum. The vector $\mathbf{n}_3$ is the unit vector in the direction of the third coordinate axis, which we choose to be identical with the axis of rotation. We also define a new coordinate $\mathbf{p}' = \mathbf{p} - m\omega\mathbf{n}_3 \times \mathbf{x}$, which is shifted momentum. As we will see later, the shift has no effect during integration. Compared to the Hamiltonian of a free particle, there is also a potential term which decreases with increasing distance from the axis. This potential represents the centrifugal force exerted on the particles in a corotating system. The variable r is the distance from the axis of rotation and it satisfies $r^2 = x_1^2 + x_2^2$.

Since this is a system with constant temperature, we describe it using a canonical statistical ensemble. We divide the tube into small layers with thickness Δr, perpendicular to the axis of the tube. We will focus on one layer, which is at a distance r from the axis of rotation. The one-particle partition function is

$$Z_1(r) = \frac{1}{h^3}\int \mathrm{e}^{-\beta H_1(X_1)}\,\mathrm{d}X_1 = \frac{1}{h^3}\left(\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\beta p'^2}{2m}}\,\mathrm{d}p'\right)^3 S\Delta r \mathrm{e}^{-\frac{\beta m\omega^2 r^2}{2}} = \frac{V}{\kappa^3}\mathrm{e}^{-\frac{\beta m\omega^2 r^2}{2}}\,,$$

where the integral is over all possible states X_1 of one particle, $\beta = \frac{1}{k_\mathrm{B}T}$ and h is Planck's constant. We used an auxiliary constant

$$\kappa = h\sqrt{\frac{2\pi m}{\beta}}$$

and denoted the volume of a layer by $V = S\Delta r$. Since we are working with an ideal gas, the particles do not interact with each other and therefore we can write the total partition function as

$$Z(r) = \frac{Z_1(r)^{N(r)}}{N(r)!}\,,$$

where $N(r)$ is the number of particles in the given layer. From statistical physics, we know that the result is connected to Helmholtz free energy $F = -\frac{1}{\beta}\ln Z$ and we also know that the chemical potential $\mu(r)$ can be calculated as the derivative of F with respect to the number of particles N. We can write

$$\mu(r) = \frac{\partial F}{\partial N} = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial N} = -\frac{1}{\beta}\left(\ln\frac{\mathrm{e}Z_1}{N} - \frac{\mathrm{e}Z_1}{N}\right) \approx \frac{1}{\beta}\ln\frac{N}{\mathrm{e}Z_1} = \frac{1}{\beta}\ln\left(\frac{N\kappa^3}{\mathrm{e}V}\mathrm{e}^{-\frac{\beta m\omega^2 r^2}{2}}\right)\,,$$

where we used the approximation

$$N! \approx \left(\frac{N}{\mathrm{e}}\right)^N$$

and neglected the term $\frac{\mathrm{e}Z_1}{N}$, because it is zero in the thermodynamic limit. Now, we use the fact that the equilibrium between individual layers requires $\mu(r) = \mu_0$ to be constant. This means that the whole formula inside the logarithm must be constant as well. If we denote the volume density of particles $n(r) = \frac{N}{V}$, we obtain

$$\frac{N\kappa^3}{\mathrm{e}V}\mathrm{e}^{-\frac{\beta m\omega^2 r^2}{2}} = \mathrm{const}\,,$$

$$n(r) = n_0\mathrm{e}^{\frac{\beta m\omega^2 r^2}{2}}\,,$$

where n_0 is the volume density of particles at the axis of rotation.

The remaining task is to calculate the position of the center of mass

$$T_r = \frac{\int_0^{r_0} r n(r)\,\mathrm{d}r}{\int_0^{r_0} n(r)\,\mathrm{d}r} = \frac{\int_0^{r_0} r\mathrm{e}^{k^2 r^2}\,\mathrm{d}r}{\int_0^{r_0} \mathrm{e}^{k^2 r^2}\,\mathrm{d}r}\,,$$

where $k = \sqrt{\frac{\beta m \omega^2}{2}}$. We obtain the same result as in the case above.

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Problem FoL.49 ... falling impulse 9 points

There is a thread with a length $l = 1.00\,\mathrm{m}$ and a total mass $m_\mathrm{v} = 1.00\,\mathrm{g}$ hanging from the ceiling. On its bottom end, a small weight with unknown mass m is hung. We send an impulse down the thread from the top and simultaneously drop a point mass from the ceiling. We notice two things:

1. *the impulse is traveling along the thread with a constant speed,*
2. *the impulse reaches the bottom end of the thread at the same time as the point mass.*

What is the mass of the little weight? Everything is taking place in homogeneous gravity with acceleration $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. *Jirka was unable to figure out how this idea struck his mind.*

Let's use x as a coordinate of distance measured from the top of the thread. The bottom end of the thread is at $x = l$. The impulse is traveling along the thread with speed

$$v(x) = \sqrt{\frac{F(x)}{\lambda(x)}} = \mathrm{const},$$

where $F(x)$ is the force of tension within the thread at a distance x and $\lambda(x)$ is the linear density of the thread at the same point. We know that the speed of the impulse is constant along the whole length of the thread. We can find the force $F(x)$ by integrating the infinitesimal contributions from pieces of thread hanging between x and l

$$F(x) = \left(m + \int_x^l \lambda(t)\,\mathrm{d}t\right) g\,.$$

We express the force from the first equation and substitute into the second one

$$\frac{v^2}{g}\lambda(x) = m + \int_x^l \lambda(t)\,\mathrm{d}t = m + L(l) - L(x)\,, \tag{6}$$

where L is the indefinite integral of λ. We differentiate this equation by x and obtain the differential equation

$$\frac{v^2}{g}\lambda' = -\lambda\,,$$

solvable by the method of separation of variables. We get

$$\lambda = \lambda_0 \mathrm{e}^{-\frac{g}{v^2}x}\,,$$

where λ_0 is an unknown constant which we need to infer from boundary conditions. We can see that the linear density of the thread decreases exponentially with length. The total mass of the thread is

$$m_\mathrm{t} = \int_0^l \lambda_0 \mathrm{e}^{-\frac{g}{v^2}t}\,\mathrm{d}t = \frac{\lambda_0 v^2}{g}\left(1-\mathrm{e}^{-\frac{gl}{v^2}}\right),$$
$$\lambda_0 = \frac{m_\mathrm{t} g}{v^2\left(1-\mathrm{e}^{-\frac{gl}{v^2}}\right)}.$$

In the equation (6), we set $x = 0$ and express the mass of the little weight

$$m = \frac{v^2}{g}\lambda_0 - \int_0^l \lambda\,\mathrm{d}x\,.$$

We've already calculated this integral once, the result is m_t. We substitute for λ_0 and get

$$m = \frac{v^2}{g}\cdot\frac{m_\mathrm{t} g}{v^2\left(1-\mathrm{e}^{-\frac{gl}{v^2}}\right)} - m_\mathrm{t} = m_\mathrm{t}\left(\frac{1}{1-\mathrm{e}^{-\frac{gl}{v^2}}}-1\right) = \frac{m_\mathrm{t}}{\mathrm{e}^{\frac{gl}{v^2}}-1}\,.$$

Now we can use the information that the impulse reaches the end in the same time t as the point mass falling from height l. We have

$$l = vt\,,$$
$$l = \frac{1}{2}gt^2\,,$$

but the velocity v is not the final velocity of the point mass, but the speed of the impulse, i.e. the mean velocity of the point mass during fall. From that follows

$$v^2 = \frac{gl}{2}\,.$$

Putting everything together, we get

$$m = \frac{m_\mathrm{t}}{\mathrm{e}^2-1} \doteq 0.157\,\mathrm{g}.$$

The mass of the little weight is roughly 0.157 g.

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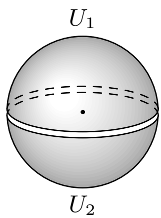
Problem FoL.50 ... a problem with potential

3 points

Lego took two hemispherical shells with the same radii and connected them in a non-conducting way such that together they formed a sphere. Then he set a potential $U_1 = 100\,\mathrm{V}$ on one hemisphere and a potential $U_2 = -100\,\mathrm{V}$ on the other hemisphere. What is the potential in the middle of the sphere?

Lego knew that he has a potential...



The problem can be solved using Laplace's equation, of course. However, let's wait with that for now...

Let's denote the desired potential in the center of the sphere by φ. If we exchange U_1 and U_2, the signs of all potentials given in the problem change. Therefore, the signs of all potentials in the solution change as well and φ changes to $-\varphi$ (we can also look at it as if there were aliens who define the signs of electric charge in the exactly opposite way; this would be the problem they would need to solve).

However, if we exchange the potentials on the spherical shell, it is also the same as if we only turned the sphere over (or if we looked at it from the opposite side). And that cannot have any impact on the potential in its center. We get the equation $\varphi = -\varphi$, which has only one solution, 0.

Therefore, the potential in the center of the sphere is $\varphi = 0\,\mathrm{V}$.

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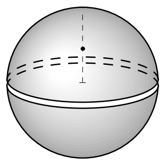
Problem FoL.51 ... another problem with potential

9 points

Lego took two hemispherical shells with the same radii and connected them in a non-conducting way such that together they formed a sphere. Then he set a potential $U_1 = 100\,\mathrm{V}$ on one hemisphere and a potential $U_2 = -100\,\mathrm{V}$ on the other hemisphere. Find the potential in the middle of a straight line connecting the centre of the sphere and the apex of the positively charged hemisphere (depicted in the figure)?

... and he also thought that it is greater than 0.



Numerical solution

The electric potential everywhere except for the regions with non-zero charge (therefore everywhere inside of the sphere) satisfies Laplace's equation

$$\Delta\Phi = 0\,.$$

For some basic idea, it is enough to realise that the Laplace operator is a sum of second derivatives in all (three) directions. In theory, we could discretize the sphere as a 3D grid. Then, using differentiation of the Laplace equation, we would obtain that the potential at each point is equal to the arithmetic mean of potentials at the neighbouring 6 points, and could let the computer iterate the given formula (with fixed potential on the border) until it converges. However, to obtain the result with the required precision, the calculation would be very time-consuming (without an external server, it could even last longer than our competition). Therefore, as usual in electrostatics problems, we use symmetry. The Laplace operator in spherical coordinates equals

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\Phi\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}\,.$$

If we choose the z-axis as identical with the symmetry axis of the problem, the potential does not depend on the φ coordinate (written as $\Phi = \Phi(r,\theta)$), i.e. the third term is zero. Most importantly, we can then discretize the problem using two variables only. Therefore, we have a 2D grid where each point is described with two indices i, j and its potential $\Phi(r,\theta) = \Phi(ih_r, jh_\theta) = f(i,j)$, where we defined a function f for clarity; h_r, h_θ are constants denoting the length of one step in the direction of r or θ. If we index our points starting from 0, then $h_r = R/i_{\max}$ and $h_\theta = \pi/2j_{\max}$, where R is the radius of the sphere, $i_{\max}$ is the largest index i and $j_{\max}$ is the largest index j, because we know from the solution of the previous problem that the potential in the plane connecting both hemispheres is everywhere equal to 0. Then we only need to find the potential inside the hemisphere which is interesting for us, i.e. where $\theta \in \langle 0, \pi/2\rangle$.

In the direction of increasing r, we have discrete points with spacing h_r. The derivative of fr at a point i, j is

$$[f(i,j)r(i)]' = \frac{f(i+1,j)r(i+1) - f(i,j)r(i)}{h_r}\,,$$

where we used only the definition of a derivative and omitted the detail that h shall be infinitely small. To be precise, we have written the so-called forward derivative, which somewhat describes the derivative of f in the region between i and $i+1$. It is useful to keep this in mind. The second derivative is the derivative of the derivative of f. Let us think about it for a while. Instead of using two forward derivatives, which would mean comparing the derivative between i and $i+1$ with the one between $i+1$ and $i+2$ (this does not seem to be the correct way to express the second derivative in the point i), we use the backward derivative (symbolically described as $f'(i) = (f(i) - f(i-1))/h$, which is, of course, only another first derivative) instead. Next, we plug in $r(i) = ih_r$ and calculate

$$\begin{aligned}\frac{1}{r}\frac{\partial^2}{\partial r^2}(fr) &= \frac{1}{r(i)}\frac{[f(i,j)r(i)]' - [(f(i-1,j)r(i-1)]'}{h_r} = \\ &= \frac{f(i+1,j)(i+1) - 2f(i,j) + f(i-1,j)(i-1)}{ih_r^2}\,.\end{aligned}$$

Similarly (with slightly more difficulty), we differentiate the second term

$$\begin{aligned}&\frac{1}{r^2(i)\sin\theta(j)}\frac{\partial}{\partial\theta}\left(\sin\theta(j)\frac{\partial f}{\partial\theta(j)}\right) = \\ &= \frac{\sin\left[\left(j+\frac{1}{2}\right)h_\theta\right](f(i,j+1) - f(i,j)) - \sin\left[\left(j-\frac{1}{2}\right)h_\theta\right][f(i,j) - f(i,j-1)]}{\sin(jh_\theta)\, i^2h_r^2h_\theta^2}\,.\end{aligned}$$

The expression $\sin\left[\left(j+\frac{1}{2}\right)h_\theta\right]$ may look quite weird, but it results from our desire to get the value of the second derivative "exactly" between j and $j+1$.

The remaining task is to use the fact that the sum of these two terms equals 0 and express

$$\begin{aligned}f(i,j) = &\frac{\sin(jh_\theta)\, ih_\theta^2\,[f(i-1,j)\,(i-1) + f(i+1,j)\,(i+1)]}{2i^2h_\theta^2\sin(jh_\theta) + \sin\left[\left(j+\frac{1}{2}\right)h_\theta\right] + \sin\left[\left(j-\frac{1}{2}\right)h_\theta\right]} + \\ &+ \frac{f(i,j-1)\sin\left[\left(j-\frac{1}{2}\right)h_\theta\right] + \sin\left[\left(j+\frac{1}{2}\right)h_\theta\right]f(i,j+1)}{2i^2h_\theta^2\sin(jh_\theta) + \sin\left[\left(j+\frac{1}{2}\right)h_\theta\right] + \sin\left[\left(j-\frac{1}{2}\right)h_\theta\right]}\,.\end{aligned}$$

This holds for the potential at all points except these with non-zero charge. Such points are on the hemispheres. However, the potential on the hemispheres is given by the task, so these are our boundary conditions.

We discretize the space into a 2D grid, pass through all its points and calculate the formula given above for each of them. We repeat this over and over again until $f(i,j)$ converges. The code in Python is attached.

```
import numpy as np
Fi=np.zeros((101,101))
Fi2=np.zeros((101,101))
h=np.pi/200
b=True
eps=10**(-3)
F=100
while b:
   b=False
   for i in range(101):
     for j in range(101):
       if (i==100):
         Fi2[i,j]=F
       elif (i==0):
         Fi2[i,j]=0
       elif (j==100):
         Fi2[i,j]=0
       elif (j==0):
         Fi2[i,j]=(np.sin(j*h)*i*h*h*(Fi[i-1,j]*(i-1)+Fi[i+1,j]*(i+1))
+2*np.sin((j+0.5)*h)*Fi[i,j+1])/(2*i*i*h*h*np.sin(j*h)+2*np.sin((j+0.5)*h))
         if abs(Fi2[i,j]-Fi[i,j])>eps:
           b=True
       else:
         Fi2[i,j]=(np.sin(j*h)*i*h*h*(Fi[i-1,j]*(i-1)+Fi[i+1,j]*(i+1))+
Fi[i,j-1]*np.sin((j-0.5)*h)+np.sin((j+0.5)*h)*Fi[i,j+1])/
(2*i*i*h*h*np.sin(j*h)+np.sin((j+0.5)*h)+np.sin((j-0.5)*h))
         if abs(Fi2[i,j]-Fi[i,j])>eps:
           b=True
   for x in range(101):
     for y in range(101):
       Fi[x,y]=Fi2[x,y]
print(Fi[50,100])
```

The result after rounding is 66 V.

Analytical solution

Let us use the general solution of Laplace's equation $\Delta\Phi = 0$ in spherical coordinates

$$\Phi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{ml}r^{l} + B_{lm}r^{-l-1}\right)Y_{lm}(\theta,\varphi)\,,$$

where $Y_{lm}(\theta,\varphi)$ are spherical harmonics and A_{ml} and B_{ml} are constants determined from the initial conditions. We will choose the coordinates in such a way that the ray $\theta = 0$ intersects the apex of one hemisphere. In such a case, the coordinate φ characterizes the azimuthal symmetry of the problem and therefore the potential is independent from it, which means that only components with $m = 0$ remain in the sum and the solution with azimuthal symmetry satisfies

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta)\,,$$

where $P_l(x)$ are Legendre polynomials. We look for potential inside the ball with radius R, which does not diverge in the centre (for $r = 0$), therefore $B_l = 0$ for all l. The problem has simplified into

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)\,. \tag{7}$$

The potential V is specified on the sphere $r = R$, which leads to

$$V(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \begin{cases} U_0 & \theta \in \left\langle 0, \frac{\pi}{2}\right) \\ -U_0 & \theta \in \left(\frac{\pi}{2}, \pi\right\rangle \end{cases}. \tag{8}$$

Our objective is to find constants A_l such that the equation holds for each $\theta \in \langle 0, \pi\rangle$. We use orthogonality of Legendre polynomials given by equation

$$\int_{-1}^{1} P_l(x)P_k(x)\,\mathrm{d}x = \frac{2}{2l+1}\delta_{lk}\,.$$

We multiply the equation (8) by $P_k(\cos\theta)$ and integrate, which gives

$$\frac{2A_l R^l}{2l+1} = \int_{-1}^{1} V(\theta)P_l(\cos\theta)\,\mathrm{d}\cos\theta\,,$$

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V(\theta)P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta\,. \tag{9}$$

In the next step, we calculate the integral

$$\int_0^{\pi} V(\theta)P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta = \int_{-1}^{0} \left(-U_0\right)P_l(x)\,\mathrm{d}x + \int_0^1 U_0 P_l(x)\,\mathrm{d}x =$$

$$= U_0\left(-\int_0^1 P_l(-x)\,\mathrm{d}x + \int_0^1 P_l(x)\,\mathrm{d}x\right) = U_0\left(1-(-1)^l\right)\int_0^1 P_l(x)\,\mathrm{d}x\,. \tag{10}$$

For even l, $A_l = 0$ holds, and for odd l we need to calculate the last integral. To do this, we use several properties of Legendre polynomials, namely the knowledge of the values on the borders of the interval of integration $P_l(1) = 1$, $P_l(0) = \frac{(-1)^{\frac{l}{2}}}{2^l}\binom{l}{\frac{l}{2}}$ for even l and the integral formula for Legendre polynomials

$$\int P_l(x) = \frac{P_{l+1}(x) - P_{l-1}(x)}{2l+1}\,.$$

Using these properties we get

$$
\begin{aligned}
\int_0^1 P_l(x)\,\mathrm{d}x &= \frac{P_{l+1}(1) - P_{l-1}(1)}{2l+1} - \frac{P_{l+1}(0) - P_{l-1}(0)}{2l+1} = \\
&= -\frac{1}{2l+1}\left(\frac{(-1)^{\frac{l+1}{2}}}{2^{l+1}}\binom{l+1}{\frac{l+1}{2}} - \frac{(-1)^{\frac{l-1}{2}}}{2^{l-1}}\binom{l-1}{\frac{l-1}{2}}\right) = \\
&= \frac{(-1)^{\frac{l-1}{2}}}{(2l+1)2^{l-1}}\left(\frac{1}{2^2}\binom{l+1}{\frac{l+1}{2}} + \binom{l-1}{\frac{l-1}{2}}\right) = \\
&= \frac{(-1)^{\frac{l-1}{2}}}{2^{l+1}}\frac{(l-1)(l+1)(l-2)!}{\left(\left(\frac{l+1}{2}\right)!\right)^2} = \frac{(-1)^{\frac{l-1}{2}}}{2^{l+1}l}\binom{l+1}{\frac{l+1}{2}}.
\end{aligned}
$$

We plug this result into (10) and then into (9) and (7), which lets us explicitly express the potential inside the sphere using Legendre polynomials as

$$
\begin{aligned}
\Phi(r,\theta) &= \sum_{l=0}^{\infty}\frac{2l+1}{2R^l}U_0\left(1-(-1)^l\right)\frac{(-1)^{\frac{l-1}{2}}}{2^{l+1}l}\binom{l+1}{\frac{l+1}{2}}r^l P_l(\cos\theta) \\
&= U_0\sum_{l\,\mathrm{odd}}\frac{(-1)^{\frac{l-1}{2}}(2l+1)}{2^{l+1}l}\binom{l+1}{\frac{l+1}{2}}\frac{r^l}{R^l}P_l(\cos\theta) \\
&= U_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+2}}\frac{4n+3}{2n+1}\binom{2n+2}{n+1}\frac{r^{2n+1}}{R^{2n+1}}P_{2n+1}(\cos\theta) \\
&= U_0\sum_{n=0}^{\infty}\frac{(-1)^n(4n+3)}{2^{2n+1}(n+1)}\binom{2n}{n}\frac{r^{2n+1}}{R^{2n+1}}P_{2n+1}(\cos\theta)\,.
\end{aligned}
$$

Now we plug in the coordinates of the point where we want to calculate the potential, which are $\theta = 0$, $r = \frac{R}{2}$, and get

$$
\begin{aligned}
\Phi\left(\frac{R}{2},0\right) &= U_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}}\frac{4n+3}{n+1}\binom{2n}{n}\frac{1}{2^{2n+1}} \\
&= U_0\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{2n+1}}\frac{4n+3}{n+1}\binom{2n}{n} \\
&= U_0\left(2-\frac{3}{\sqrt{5}}\right) \doteq 0.658\,U_0\,.
\end{aligned}
$$

Halfway from the centre to the positively charged apex of the hemisphere, the potential is $\Phi \doteq 66\,\mathrm{V}$. It is enough to evaluate the first three terms of the series to obtain the result with sufficient precision, i.e. the first two significant figures.

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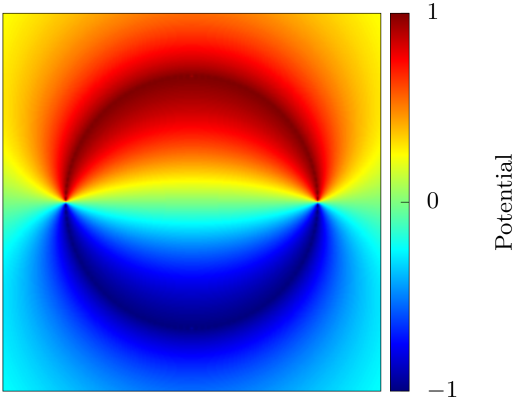

Fig. 1: Visualization of the potential between the hemispheres (normalized).

Problem M.1 ... at the train station 3 points

Legolas was trying to catch a train. He was running with a speed $v = 3.0\,\mathrm{m\cdot s^{-1}}$ when he noticed that one of his shoelaces was untied. Therefore, he stopped and spent $t = 10.0\,\mathrm{s}$ tying it. He then jumped on an escalator, which was moving with a speed $u = 1.0\,\mathrm{m\cdot s^{-1}}$ (on which he ran with speed v as well). Suddenly, he facepalmed, realizing he could have made one more step, tied the shoelace on the escalator and saved some time. But how much exactly?

You do not need to know the exact length of the escalator – it is sufficient to know that it was long enough for Legolas to manage to tie his shoelace on it. *Lego missed the train.*

We're interested in the difference of time between these two scenarios. In the first one, Legolas stops moving for a time t and then travels on the escalator with velocity $u + v$. In the second scenario, he first travels on the escalator for the time t with velocity u and then he traverses the rest of the escalator with velocity $u + v$ (the problem statement clarifies that there is some distance left).

The velocity of traversing the second part is the same in both scenarios. This means we are only interested in the time difference after traversing the first part of the escalator, which is the distance Legolas covered while tying his shoelaces in the second scenario. The distance of this point from the entrance to the escalator is $l = ut$.

In the second scenario, he reaches this point in time t. In the first one, he's first stationary for time t and then covers the distance l in time

$$t_l = \frac{l}{u+v} = t\frac{u}{u+v}\,.$$

Hence, the time difference is

$$\Delta t = (t + t_l) - t = t_l = t\frac{u}{u+v} = 2.5\,\mathrm{s}\,.$$

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Problem M.2 ... at the airport
3 points

Dano and Danka are in a hurry to catch a plane to Mexico, so they both run on a moving walkway (in the direction of its movement). Both of them have the same step length, which is exactly 1 m. *Dano runs twice as fast as Danka and he has to take 28 steps to run from one end of the walkway to the other end, while Danka needs to take just 21 steps. How long is the walkway?* *Legolas wanted to catch a plane.*

We denote the velocity of the walkway by v and the frequency of Danka's steps by f (the step length is the same, so Dano's frequency is $2f$). From Dano's point of view, the length of the walkway is $s_{\mathrm{Dano}} + s_{\mathrm{ww}}$, where s_{Dano} is the distance Dano walked on the walkway, that is $s_{\mathrm{Dano}} = 28\,\mathrm{m}$, and s_{ww} is the distance the walkway traveled while Dano was standing on it. Dano spent the time $28/(2f)$ on the belt, so the length of the walkway is $s_{\mathrm{Dano}} + 14v/f$. We can do the same calculation for Danka, to obtain the length of the walkway as $s_{\mathrm{Danka}} + 21v/f$, where $s_{\mathrm{Danka}} = 21\,\mathrm{m}$. Comparing these two relationships, we obtain $v/f = 1\,\mathrm{m}$, and substituting into (either of) the formulas for the length of the walkway, we determine the length of the walkway as 42.0 m.

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Problem M.3 ... under the ground
3 points

Matěj likes to travel by metro. One day, he got off at a station which was $h = 50\,\mathrm{m}$ *deep under the surface. He walked upwards with a speed* $v = 1\,\mathrm{m\cdot s^{-1}}$ *on an escalator, which itself moves upwards with a speed* $u = 1.5\,\mathrm{m\cdot s^{-1}}$. *What work has Matěj done by walking upwards? Assume that his mass is* $m = 60\,\mathrm{kg}$. *Matěj likes escalators and the metro.*

The total work done was $W = mgh$. Part of it was done by Matěj by walking upwards, part was done by the escalator. Because the work is directly proportional to the traveled height h, it will be split in the same ratio as the heights covered by each of them. The time of motion is the same for both Matěj and the escalator and they move in the same direction, so the height is directly proportional to the speeds of movement. The height covered by the effort of the person is

$$h_{\mathrm{M}} = h\frac{v}{v+u}\,,$$

whilst the escalator lifts him by the height

$$h_{\mathrm{e}} = h\frac{u}{v+u}\,.$$

From this we can calculate the work done by Matěj as

$$W_{\mathrm{M}} = mgh_{\mathrm{M}} = mgh\frac{v}{v+u} \doteq 11\,800\,\mathrm{J}\,.$$

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Problem M.4 ... in the shopping mall 3 points

You have traveled by train, by metro and by plane to visit a shopping mall, which is, of course, equipped with many escalators. On a switched-on escalator moving upwards, the time it takes you to walk up and down again is $t_{\mathrm{on}} = 50\,\mathrm{s}$. On a switched-off escalator, it takes $t_{\mathrm{off}} = 30\,\mathrm{s}$. What is the speed of the stairs of a switched-on escalator, if you walk with a speed $v_{\mathrm{u}} = 1.0\,\mathrm{m{\cdot}s^{-1}}$ upwards and $v_{\mathrm{d}} = 2.0\,\mathrm{m{\cdot}s^{-1}}$ downwards?

Legolas runs up and down escalators in metro stations.

We can find the length of the escalator from the case when it is switched off as

$$t_{\mathrm{off}} = \frac{l}{v_{\mathrm{u}}} + \frac{l}{v_{\mathrm{d}}} \quad \Rightarrow \quad l = \frac{t_{\mathrm{off}} v_{\mathrm{u}} v_{\mathrm{d}}}{v_{\mathrm{u}} + v_{\mathrm{d}}}\,.$$

For the switched-on escalator, we label its speed as u and we have

$$t_{\mathrm{on}} = \frac{l}{v_{\mathrm{u}} + u} + \frac{l}{v_{\mathrm{d}} - u} \quad \Rightarrow \quad l = \frac{t_{\mathrm{on}}}{v_{\mathrm{u}} + v_{\mathrm{d}}}\left(v_{\mathrm{u}} + u\right)\left(v_{\mathrm{d}} - u\right)\,.$$

We can therefore construct an equation

$$\begin{aligned} \frac{t_{\mathrm{on}}}{v_{\mathrm{u}} + v_{\mathrm{d}}}\left(v_{\mathrm{u}} + u\right)\left(v_{\mathrm{d}} - u\right) &= \frac{t_{\mathrm{off}} v_{\mathrm{u}} v_{\mathrm{d}}}{v_{\mathrm{u}} + v_{\mathrm{d}}}\,,\\ v_{\mathrm{u}} v_{\mathrm{d}} + u\left(v_{\mathrm{d}} - v_{\mathrm{u}}\right) - u^2 &= \frac{t_{\mathrm{off}} v_{\mathrm{u}} v_{\mathrm{d}}}{t_{\mathrm{on}}}\,,\\ u^2 - \left(v_{\mathrm{d}} - v_{\mathrm{u}}\right) u + \left(\frac{t_{\mathrm{off}}}{t_{\mathrm{on}}} - 1\right) v_{\mathrm{u}} v_{\mathrm{d}} &= 0\,. \end{aligned}$$

We've obtained a quadratic formula for u, the solutions of which are

$$u = \frac{v_{\mathrm{d}} - v_{\mathrm{u}} \pm \sqrt{(v_{\mathrm{d}} + v_{\mathrm{u}})^2 - 4 v_{\mathrm{d}} v_{\mathrm{u}} \frac{t_{\mathrm{off}}}{t_{\mathrm{on}}}}}{2}\,,$$

and we are interested in the solution with the plus sign (the one with the minus sign is negative), which is approximately $u \doteq 1.5\,\mathrm{m{\cdot}s^{-1}}$.

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Problem E.1 ... statistical 1 - intro 3 points

In statistical physics, we can describe a system of particles enclosed in an unchanging volume. We call this a cannonical system and the probability that we find the system in a certain state n follows Boltzmann's Law

$$p_n = \frac{1}{Z} \mathrm{e}^{-\frac{E_n}{k_\mathrm{B} T}} \,,$$

where E_n is the energy of the system in the state n, T is the temperature of the system and $k_\mathrm{B} = 1.381 \cdot 10^{-23}\,\mathrm{m^2 \cdot kg \cdot s^{-2} \cdot K^{-1}}$ is the Boltzmann constant. The value of Z is chosen in such a way that the sum of p_n over all the states equals one. Typically, there is a large number of states a system can reside in, but in this simple problem, let's work with a system with only three distinguishable states ($n = 1, 2, 3$) with energies $E_1 = 1.00 \cdot 10^{-20}\,\mathrm{J}$, $E_2 = 2E_1$ and $E_3 = 3E_1$. Determine the value of Z at a temperature $T_0 = 275\,\mathrm{K}$.

Matěj missed statistical physics among FYKOS problems.

Let's start with the fact that the sum of all probabilities is equal to 1

$$1 = p_1 + p_2 + p_3 = \frac{1}{Z} \left(\mathrm{e}^{-\frac{E_1}{k_\mathrm{B} T}} + \mathrm{e}^{-\frac{E_2}{k_\mathrm{B} T}} + \mathrm{e}^{-\frac{E_3}{k_\mathrm{B} T}} \right) \,,$$

$$Z = \mathrm{e}^{-\frac{E_1}{k_\mathrm{B} T}} + \mathrm{e}^{-\frac{E_2}{k_\mathrm{B} T}} + \mathrm{e}^{-\frac{E_3}{k_\mathrm{B} T}} \doteq 0.077\,4 \,.$$

The specific value of Z usually does not have any physical meaning. It is simply a normalizing constant. Only its derivatives will be physically relevant, as we shall see in the subsequent problems.

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Problem E.2 ... statistical 2 - basics 3 points

Now, let's examine a canonical system which has only two possible states. The difference of energies between these two states is exactly $\Delta E = 10^{-20}\,\mathrm{J}$. For example, you can imagine a molecule that exists either in its ground state or an excited state, but this description works for any general two-state system. What is the probability of finding the system in the state with higher energy if the temperature of the system is $T_0 = 275\,\mathrm{K}$?

Note: use knowledge from the previous problem. *Because the intro was not enough.*

According to the problem statement, we have only two available states with energies E_1 and $E_2 = E_1 + \Delta E$. In this case, we can easily plug in all (i.e. both) cases into the condition that the sum of all probabilities equals 1, so we get

$$p_1 + p_2 = 1 \,,$$

$$\frac{1}{Z} \mathrm{e}^{-\frac{E_1}{k_\mathrm{B} T}} + \frac{1}{Z} \mathrm{e}^{-\frac{E_1 + \Delta E}{k_\mathrm{B} T}} = 1 \,,$$

$$1 + \mathrm{e}^{-\frac{\Delta E}{k_\mathrm{B} T}} = Z \mathrm{e}^{\frac{E_1}{k_\mathrm{B} T}} \,.$$

Substituting into Boltzmann's law we obtain

$$p_2 = \frac{1}{Z e^{\frac{E_1}{k_B T}}} e^{-\frac{\Delta E}{k_B T}} = \frac{e^{-\frac{\Delta E}{k_B T}}}{1 + e^{-\frac{\Delta E}{k_\mathrm{B} T}}} = \frac{1}{1 + e^{\frac{\Delta E}{k_\mathrm{B} T}}} \doteq 0.0670\,.$$

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Problem E.3 ... statistical 3 - the quantum way — 4 points

Earlier, you had to express Z, which you can calculate using

$$Z = \sum_n e^{-\frac{E_n}{k_\mathrm{B} T}}\,,$$

where the sum goes over all possible states (such as the two in the previous problem). Then you had to use Boltzmann's Law to calculate probabilities. The sum Z is called the partition function.

Now let's consider a more complicated system - the quantum harmonic oscillator, which can be found in many different states. The energy of the n-th state is $\left(n + \frac{1}{2}\right)\hbar\omega$, where $n \in \mathbb{N}_0$ and $\hbar\omega = 10^{-21}$ J is the parameter of the oscillator. Determine the probability that the system is in the ground state ($n = 0$). The temperature of the system is $T_0 = 275\,\mathrm{K}$.

Hint $\sum\limits_{n=0}^{\infty} q^n = \frac{1}{1-q}$ *for $q \in (0, 1)$.* *This is where the fun begins.*

We calculate the partition function

$$Z = \sum_{n=0}^{\infty} e^{-\frac{\left(n+\frac{1}{2}\right)\hbar\omega}{k_\mathrm{B} T}} = e^{-\frac{\hbar\omega}{2k_\mathrm{B} T}} \sum_{n=0}^{\infty} \left(e^{-\frac{\hbar\omega}{k_\mathrm{B} T}}\right)^n = \frac{e^{-\frac{\hbar\omega}{2k_\mathrm{B} T}}}{1 - e^{-\frac{\hbar\omega}{k_\mathrm{B} T}}}$$

and substitute it into Boltzmann's law

$$p_0 = \frac{1 - e^{-\frac{\hbar\omega}{k_\mathrm{B} T}}}{e^{-\frac{\hbar\omega}{2k_\mathrm{B} T}}} e^{-\frac{\hbar\omega}{2k_\mathrm{B} T}} = 1 - e^{-\frac{\hbar\omega}{k_\mathrm{B} T}} \doteq 0.2315\,.$$

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Problem E.4 ... statistical 4 - the second quantum one — 4 points

In the previous problem, you calculated the partition function of a quantum harmonic oscillator with energies $\left(n + \frac{1}{2}\right)\hbar\omega$, where $\hbar\omega = 10^{-21}$ J. It's actually possible to obtain all the different thermodynamic properties of the system just using the partition function. For example, the internal energy of the system $\overline{E}$ (the expected value of energy) can be calculated as

$$\overline{E} = k_\mathrm{B} T^2 \frac{\partial}{\partial T} \ln\left(Z(T)\right).$$

Determine the internal energy of the quantum harmonic oscillator from the previous problem in the units of $\hbar\omega$. The temperature of the system is $T_0 = 275\,\mathrm{K}$. This was supposed to be a problem with a second partial derivative, but it would probably be too hard.

We differentiate the partition function and get

$$\begin{aligned}\overline{E} &= k_\mathrm{B}T^2 \frac{\partial}{\partial T}\ln\left(Z(T)\right) = k_\mathrm{B}T^2\frac{1}{Z}\frac{\partial Z}{\partial T} = \\ &= k_\mathrm{B}T^2\frac{1}{Z}\frac{\frac{\hbar\omega}{2k_\mathrm{B}T^2}\mathrm{e}^{-\frac{\hbar\omega}{2k_\mathrm{B}T}}\left(1-\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}\right)+\frac{\hbar\omega}{k_\mathrm{B}T^2}\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}\mathrm{e}^{-\frac{\hbar\omega}{2k_\mathrm{B}T}}}{\left(1-\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}\right)^2} = \\ &= \hbar\omega\frac{\frac{1}{2}\left(1-\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}\right)+\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}}{\left(1-\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}\right)} = \frac{1}{2}\hbar\omega\frac{1+\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}}{1-\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}} = \frac{1}{2}\hbar\omega\coth\frac{\hbar\omega}{2k_\mathrm{B}T} \doteq 3.8197\,\hbar\omega\,.\end{aligned}$$

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Problem X.1 ... the strongman and the weight — 3 points

Imagine a $m = 10\,\mathrm{kg}$ weight attached to the end of a $d = 1.0\,\mathrm{m}$ long (and very light) rod. How many times bigger is the force our biceps needs to exert to hold the rod with the weight steady, compared to simply holding it in hand? Assume that our forearm is $l = 30\,\mathrm{cm}$ long. We hold the rod in such a way that it forms a "straight-line extension" of our forearm.

Dodo was carrying a pan full of water.

The force that the biceps must exert can be calculated from an equlibrium of forces on a lever, with the elbow joint serving as a pivot. By changing the position of the weight, we extend the length of the lever arm. We obtain the ratio of forces

$$\frac{F_2}{F_1} = \frac{l+d}{l} \doteq 4.3\,.$$

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Problem X.2 ... (not) breathing — 3 points

In a single day, approximately $V = 11\,000\,\mathrm{l}$ of air pass through human lungs. Find the average speed of air flowing through the larynx, assuming that its diameter is $d = 40\,\mathrm{mm}$.

Dod and his treacherous factors of two.

To solve this, we use the relation for volumetric flow rate of fluids Q

$$Q = Sv$$

where S is the cross-sectional area and v is the flow speed. The flow rate can be determined from the volume of air exchanged. The volume V has to flow both into the lungs and out of them in a day, so the average flow rate is $Q = 2V/1\,\text{day}$. The cross-section of the larynx has an area

$$S = \pi \left(\frac{d}{2}\right)^2 .$$

By expressing the flow speed and substituting for S and Q, we get

$$v = \frac{Q}{S} = \frac{8V}{\pi d^2 \cdot 1\,\text{day}} \doteq 0.20\,\mathrm{m{\cdot}s^{-1}} .$$

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Problem X.3 ... the pressure of a ballerina — 3 points

A ballerina that weighs $m = 50\,\mathrm{kg}$ is landing on a big toe after jumping into the height $h = 0.5\,\mathrm{m}$. At its narrowest point, the last phalanx bone of the big toe has a diameter $d = 1.3\,\mathrm{cm}$. Find the highest stress across this bone during the landing. Assume that the ballerina decelerates upon impact with a constant force for $t = 0.5\,\mathrm{s}$. *Dodo stood on one toe.*

The stress can be calculated from its definition as a force across a cross-sectional area

$$p = \frac{F}{S} = \frac{4F}{\pi d^2} .$$

The force transferred through the bone is given by the sum of the weight of the ballerina and the force used for deceleration, which can be obtained from Newton's 2nd law as the rate of change of the momentum

$$F_\mathrm{d} = \frac{mv}{t} = \frac{m\sqrt{2gh}}{t} ,$$

into which we substituted the free fall velocity right before impact.

Substituting the expression for the force into the equation for stress, we obtain

$$p = \frac{4\left(mg + \frac{m\sqrt{2gh}}{t}\right)}{\pi d^2} \doteq 6.06\,\mathrm{MPa} .$$

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Problem X.4 ... saliva — 3 points

Fykos organizers are very skillful and managed to create $100\,\mathrm{g}$ a substance such that $99\,\%$ of its mass is water. Unfortunately, after a while, some water evaporated away and now only $98\,\%$ of the sample is water. What is the current mass of the sample? *Lego likes Vsauce2.*

The key is to realize that the mass of the non-water part of the sample is conserved. In the beginning, it is $100\,\% - 99\,\% = 1\,\%$ of the total mass. Since the sample weighed $100\,\mathrm{g}$, the non-water part weighs $100\,\mathrm{g} \cdot 1\,\% = 1\,\mathrm{g}$.

After drying out, the non-water part makes up $100\,\% - 98\,\% = 2\,\%$ of the total mass. The whole sample must then weigh $1\,\mathrm{g}/2\,\% = 50\,\mathrm{g}$.

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